Conditional and Dependent Credit Migrations in a Factor Model Copula Approach

Stefan Trück

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Conditional and Dependent Credit Migrations in a Factor Model Copula Framework

Stefan Trück*,

Macquarie University Sydney, Australia

Abstract

We review different methods for simulating credit migrations in a nonparametric and discrete or continuous-time Markov chain framework. We suggest the application of a factor model approach in combination with the use of copulas for the joint dynamics of credit rating changes. While there are several applications of copulas in credit risk for modeling joint defaults, it lacks the same interest towards modeling dependence in rating migrations. It is well-known, however, that the risk of a credit portfolio is not dependent only on the defaults but also on rating upgrades and downgrades. In a simulation study, we illustrate the effects of considering dependencies in credit migrations for an exemplary loan portfolio. Hereby, we do not only examine default or loss figures for the portfolio, but also the distribution of ratings by the end of the simulated period. Our findings illustrate quite large differences between the different approaches: not only the fact whether dependence is accounted for but also the choice of the copula affects loss figures and the distribution of ratings. Also extreme outcomes for credit migrations like they have been observed during times of a financial crisis can be modeled introducing an adequate level of dependence. The approach may be in particular helpful for scenario analysis and stress testing in credit risk management.

Key words: Credit Risk, Rating Migrations, Copulas, Credit VaR, Scenario Analysis, Financial Crisis

* Email: strueck@efs.mq.edu.au, Address: Faculty of Business and Economics, Macquarie University Sydney, Australia.
1 Introduction

Within the last decade, the market for credit related products as well as techniques for credit risk management have undergone significant changes. Especially rating based models have become very popular. On the one hand this popularity is due to the straightforwardness of the approach: the models use the rating of a company as the decisive variable when it comes to evaluate the default risk of a bond or loan. They also avoid the difficulties of structural models (Merton, 1974) like determining a company’s value and volatility. On the other hand, the new capital accord (Basel II) encourages banks to base their capital requirement for credit risk on internal or external rating systems (Basel Committee on Banking Supervision, 2001). As a result, the majority of international operating banks applies internal ratings based approaches to determine capital requirements for their loan or bond portfolios. Overall, despite some deficiencies of the current credit rating structure being revealed during the subprime mortgage crisis and the analysis for necessary improvements, see e.g. Crouhy et al. (2008), rating based models have evolved as an industry standard in the field of credit risk management. However, considering the significant and sudden changes in creditworthiness of many obligors during the financial crisis, scenario analysis techniques and models taking into account also the possibility of such extreme outcomes in rating changes for a portfolio are required.

Due to cyclicalility in the economy there are large effects on creditworthiness, default rates and migration behavior of loans or bonds (Allen and Saunders, 2003). Therefore, migration matrices are not constant through time as it is pointed out by Nickell et al. (2000); Bangia et al. (2002); Wei (2003). Further investigating the issue, Trück and Rachev (2005) show that such changes in migration or default behavior lead to substantial effects on risk figures for credit portfolios. In recent years, more effort has been made to describe the nature not only of cyclical effects, but also of the dependence of these changes in default behavior. As Frey and McNeil (2003) report, defaults and downgrades seem to happen in clusters during recessions or economic downturns. On the other hand, also during a business cycle peak, default figures are significantly lower and more rating upgrades can be observed, see e.g. Bangia et al. (2002). In particular copulas have become a very popular framework for modeling dependence of default-risky instruments. Initially suggested by Vasicek (1987); Li (2000), a substantial number of publications has been dedicated to the topic in recent years. References include Frey and McNeil (2003); Giesecke (2004); Laurent and Gregory (2005); Hull and White (2004); Mashal and Naldi (2002); Schönbucher and Schubert (2001); Schönbucher (2003), just to name a few. An excellent treatment on the issue can also be found in McNeil et al. (2005). However, while most of the authors concentrate on the modeling of joint defaults, the same interest seems to lack towards the modeling of joint
or dependent migrations in a Markov chain framework. In particular in the long run the risk of a credit portfolio will also be affected by rating upgrades or downgrades and not only direct transitions to the default state.

In this paper we will review techniques for simulation of credit migrations and provide some approaches that focus on modeling dependencies in rating transitions within a factor model combined with a copula framework. As mentioned above there is only a limited number of publications focusing on dependencies also in credit migrations. Exceptions include Hamilton et al. (2001) who apply copulas for credit migrations in an intensity based framework and use them to price credit derivatives. Further investigating the issue, Gagliardini and Gourieroux (2005a,b) present a general framework for rating dynamics, based on stochastic migration matrices. In an empirical study, using French corporate data they focus on serial correlation between migration matrices. McNeil and W¨endin (2006) use ordered categorical variables and induce dependence between migrations by means of latent risk factors. We suggest a simpler approach to either integrate the dependence structure by using correlations or copulas. We start with a factor model approach using correlations that has been used for forecasting of conditional transition matrices (Belkin et al., 1998b; Kim, 1999; Tr¨uck, 2008). In these models credit migrations are modeled as being dependent on a systematic risk factor and an idiosyncratic, firm-specific factor. The dependence in credit migrations can then be triggered by the degree of correlation with the systematic risk factor. We then extend the framework by suggesting the use of copulas for modeling the joint dynamics of credit rating changes. In particular the Gaussian and Student $t$-copula will be used to show the effects of different assumptions about the degree of dependence and the choice of the copula. Our results show that both the integration of dependence as well as the choice of the copula have substantial impacts on the Value-at-Risk and the distribution of ratings in the portfolio while the expected loss remains rather constant. Our simple but appropriate framework for simulating dependent credit migrations can be considered as particular helpful for stress tests and scenario analysis conducted by senior risk managers in the financial industry.

Section two gives a brief review on rating based modeling and the use of credit migration matrices in risk management. We further review different techniques for simulating credit migrations based on a discrete time, continuous-time parametric and a non-parametric approach. Section three then extends these techniques by considering dependent credit migrations using a factor model in combination with copulas. Section four provides empirical results on the simulation process assuming independent migrations and using different copulas for dependent migrations. Section five concludes and gives suggestions for future work.
This section is dedicated to rating-based modeling with focus on migration matrices within a discrete and continuous-time Markov chain approach. As an extension of the simple reduced form model by Fons (1994) in these models not only the event of default is considered but also the probabilities of rating changes for a company or an issued bond. Downgrades or upgrades are taken very seriously by market players to price bonds and loans, thus effecting the risk premium and the yield spreads. Information on historical defaults and rating upgrades or downgrades are provided by the major rating agencies, for example by Moody’s or Standard & Poor’s. They can also be calculated by the banks themselves for their internal loan portfolio based on an internal rating system as it is suggested in the new Basel Capital Accord (Basel Committee on Banking Supervision, 2001). In both cases the available information on a company or an obligor is summarized by a single validation of its capability to pay back debt obligations that is pronounced in form of a rating. In several of the popular industry models like CreditMetrics or CreditPortfolioView, historical transition matrices are then used to determine VaR figures for a portfolio as well as adequate bond prices.

2.1 The Markov Chain Approach

One of the most popular rating-based model in the academic literature is probably the discrete-time Markovian model by Jarrow et al. (1997) (JLT). In their seminal paper the authors suggest to model default and transition probabilities by a discrete, time-homogeneous Markov chain on a finite state space \( S = \{1, \ldots, K\} \) (Jarrow et al., 1997). The state space \( S \) represents the different rating classes. While state \( S = 1 \) denotes the best credit rating, \( K \) represents the absorbing default state. Hence, the \((K \times K)\) one-period transition matrix is denoted by:

\[
P = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1K} \\
p_{21} & p_{12} & \cdots & p_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
p_{K-1,1} & p_{K-1,2} & \cdots & p_{K-1,K} \\
0 & 0 & \cdots & 1
\end{pmatrix},
\]

where \( p_{ij} \geq 0 \) for all \( i, j, i \neq j \), and \( p_{ii} = 1 - \sum_{j \neq i}^{K} p_{ij} \) for all \( i \). The variable \( p_{ij} \) represents the actual probability of going to state \( j \) from initial rating state \( i \) in one time step.
For practical purposes Lando and Skødeberg (2002) recommend to model rating migrations via a continuous-time Markov chain using a generator matrix. Following Noris (1998) we provide the definition of a generator:

**Definition 1** A generator of a time-continuous Markov chain is given by a matrix $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq K}$ satisfying the following properties:

1. $\sum_{j=1}^{K} \lambda_{ij} = 0$ for every $i = 1, \ldots, K$;
2. $0 \leq -\lambda_{ii} \leq \infty$ for every $i = 1, \ldots, K$;
3. $\lambda_{ij} \geq 0$ for all $i, j = 1, \ldots, K$ with $i \neq j$.

Thus, a continuous-time time-homogeneous Markov chain is specified via a $K \times K$ generator matrix of the form

$$\Lambda = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1K} \\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{K-1,1} & \lambda_{K-1,2} & \cdots & \lambda_{K-1,K} \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad (2)$$

with negative diagonal elements $\lambda_{ii} = -\sum_{j \neq i}^{K} \lambda_{ij}$ for $i = 1, \ldots, K$, representing the intensities of jumping from rating state $i$ to state $j$. Further, default $K$ is again considered to be an absorbing state. Thus, based on historical default rates and empirical observed credit spreads, Jarrow et al. (1997) provide a model that can be used for various aspects in risk management or credit derivative pricing. Hereby, numerical techniques are used to adjust historical migration matrices into risk-neutral ones such that the corresponding default probabilities match observed credit spreads in bond prices.

### 2.2 Simulating Credit Migrations

In the following we will review some techniques that can be used to simulate credit migration matrices. Simulation results from migration matrices are of particular interest when it comes to calculations of Value-at-Risk or expected shortfall for a credit or loan portfolio. We will give a quick overview on different simulation techniques and present algorithms for a discrete time, continuous-time and non-parametric approach. Note that hereby we do not focus on the
idea of introducing dependent transitions yet but assume that individual loans or bonds migrate independently.

**Time Discrete Case**

In the time discrete case the simulation procedure is straightforward and can be conducted the following way: depending on the initial rating $i$ of the firm the interval $[0, 1]$ is divided into sub-intervals according to the migration probabilities $p_{ij}$ for $j = 1, ..., K$. For example, for each rating class $i$ the intervals can be determined according to the following procedure:

\[
I_{1,i} = [0, p_{i,1})
\]
\[
I_{2,i} = [p_{i,1}, p_{i,1} + p_{i,2})
\]
\[
... ...
\]
\[
I_{j,i} = \left[ \sum_{k=1}^{j-1} p_{i,k}, \sum_{k=1}^{j} p_{i,k} \right)
\]
\[
... ...
\]
\[
I_{K,i} = \left[ \sum_{k=1}^{K-1} p_{i,k}, 1 \right]
\]

Then a uniform distributed random variable $u_t$ between 0 and 1 is drawn. Depending on which sub-interval the random variable lies in, the company stays in the same rating class $i$ or migrates to rating class $j$. The migration process for a company or loan in rating class $i$ is determined by the following function $f : [0, 1] \rightarrow S$:

\[
f_{s_i} = \begin{cases} 
S_1, & \text{for } u_t \in I_{1,i} \\
S_2 & \text{for } u_t \in I_{2,i} \\
... & ...
\end{cases}
\]

\[
S_K \text{ for } u_t \in I_{K,i}
\]

If more than one time-period is considered and a migration to rating state $j$ occurs, new sub-intervals have to be calculated based on the migration probabilities $p_{jk}$ for $k = 1, ..., K$ and a new random number $u_{t+1}$ is drawn for the following period. The procedure is either going to be repeated for $t = 1, \cdots, T$ periods or terminated, if the company migrates to the absorbing default state.

**Time Continuous Case**

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Recall that a continuous-time, time-homogeneous Markov chain is specified via the a $K \times K$ generator matrix of the following form:

$$
\Lambda = \\
\begin{pmatrix}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1K} \\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{K-1,1} & \lambda_{K-1,2} & \cdots & \lambda_{K-1,K} \\
0 & 0 & \cdots & 0
\end{pmatrix},
$$

(5)

where $\lambda_{ij} \geq 0$, for all $i, j$ and $\lambda_{ii} = -\sum_{j \neq i}^{K} \lambda_{ij}$, for $i = 1, \ldots, K$. The off-diagonal elements represent the intensities of jumping from rating $i$ to rating $j$. The default state $K$ is considered to be absorbing.

In the following we will illustrate two different techniques that might be used to simulate credit migrations from a continuous-time Markov chain. The first one is applied for example in Christensen et al. (2004) or Trück and Rachev (2005).

As the waiting time for leaving state $i$ has an exponential distribution with the mean $\frac{1}{\lambda_{ii}}$, we draw an exponentially-distributed random variable $t_1$ with the density function

$$
f(t_1) = -\lambda_{ii}e^{-\lambda_{ii}t_1}
$$

for each company with initial rating $i$. Depending on the considered time horizon $T$ for $t_1 > T$, the company stays in its current class during the entire period $T$. If we get $t_1 < T$, we have to determine to which rating class the company migrates. Hence, similar to the discrete-time approach the interval $[0, 1]$ is divided into sub-intervals according to the migration intensities calculated via $\frac{\lambda_{ij}}{\lambda_{ii}}$. Then an uniform distributed random variable between 0 and 1 is drawn. Depending on which sub-interval the random variable lies in, we determine the new rating class $j$, the company migrates to. Then we draw again from an exponentially-distributed random variable $t_2$ - this time with parameter $\lambda_{jj}$ from the generator matrix. If we find that $t_1 + t_2 > T$ the considered company stays in the new rating class and the simulation is completed for this firm. If $t_1 + t_2 < T$ we have to determine the new rating class. The procedure is repeated until we get $\sum t_k > T$ or the company migrates to the absorbing default state where it will remain for the rest of the considered time period.

An alternative procedure that could be used follows an algorithm that is e.g. described in Glasserman (1992). Recall that the waiting times for leaving state $i$ to any other rating state $j$ have exponential distributions with mean
\(1/\lambda_{ij}\). Therefore we draw for each of the companies with initial rating \(i\) \(K-1\) exponentially distributed random variables \(t_{ij}\) with density functions

\[
f(t_{ij}) = \lambda_{ij}e^{\lambda_{ij}t_{ij}}
\]  

(6)

We then determine the minimum of the drawn waiting times \(t_{min_1} = \min(t_{ij})\) for \(j \neq i\). Depending on the time horizon \(T\), if \(t_{min_1} > T\) the company stays in its current rating state \(i\) for the entire period \(T\). If \(t_{min_1} < T\), the company migrates to the rating class \(j\) with the smallest drawn waiting time \(t_{ij}\). In the case of migration we again draw seven exponentially distributed random variables \(t_{jk}\) with density functions \(f(t_{jk}) = \lambda_{jk}e^{\lambda_{jk}t_{jk}}\) and determine \(t_{min_2}\). If \(t_{min_1} + t_{min_2} > T\) the company stays in the new rating class \(j\) and the simulation for this firm is completed. If \(t_{min_1} + t_{min_2} < T\), the company migrates to the rating class \(k\) with the smallest drawn waiting time \(t_{jk}\). The procedure is repeated until \(\sum_i t_{min_i} > T\) or the company migrates to default state.

**Non-parametric Approach**

Finally, we will describe the underlying idea for non-parametric simulation of credit migrations. For an empirical application, see e.g. Schuermann and Hanson (2004) or Hanson and Schuermann (2006). The authors use a non-parametric simulation procedure to estimate confidence intervals for probabilities of default. Note that for this approach the estimated discrete or continuous-time migration matrix will not be sufficient. To apply a non-parametric approach, the actual individual migrations of the loans or bonds are required. Based on the observed migrations, then for each of the rating classes a data table can be constructed containing the duration times and the rating class the corresponding company migrated to.

The simulation procedure can be described as follows: for each of the companies we draw randomly a row in the corresponding duration time data table. If the duration time \(t\) in the drawn row is greater than the considered time horizon \(T\), the company stays in the initial rating class. For \(t < T\) we have to differentiate between two cases. If the initial rating and the end rating of the drawn row coincide, the company stays of course in the initial rating. We then have to randomly draw another row from the same duration time data table, whereas the considered time horizon \(T\) has to be reduced by \(t\). If the end rating \(j\) and the initial rating \(i\) of the drawn row vary, a migration to rating \(j\) occurs. In this case we have to randomly draw a row from the duration time data table of the new rating class \(j\). The procedure is repeated until either a default occurs or the sum of the drawn duration times exceed the simulated time horizon \(T\).
The following example may help to illustrate the procedure. Table 1 gives an exemplary representation for hypothetical duration times in rating class AA. Assume that for each of the other rating classes a similar table was constructed. Hereby, also the duration records of companies which remained in the initial rating class for the considered time period were included. The simulation is then conducted according to the algorithm described above. Depending on the drawn random number the corresponding row in the table is considered to simulate the duration time in the rating class and the migration.

Overall, whether a discrete, continuous-time parametric or a non-parametric approach is chosen depends on the available data and the purpose of the conducted simulation. Lando and Skødeberg (2002) point out that one of the advantages of the continuous-time estimation technique is that it leads to more realistic non-zero estimates for probabilities of rare events, whereas the discrete time or multinomial method often leads to zero estimates for investment grade rating classes. On the other hand, as pointed out by Trück and Rachev (2005), often in the internal rating system of a bank only discrete-time historical transition matrices are reported. To benefit from the advantages of continuous-time modeling or it might then be necessary to find the correspondent generator to the discrete time transition matrix. In this case, an important issue is whether for a given discrete one-year transition matrix a corresponding so-called ‘true’ generator matrix exists, see e.g. Israel et al. (2000). Alternatively, an approximation of the generator matrix can be used. For further methods on finding the corresponding generator matrix or approximations we refer to Israel et al. (2000); Trück and Rachev (2005). As mentioned above, the non-parametric approach for simulating credit migrations is illustrated in Schuermann and Hanson (2004). Note that in this section we assumed that all individual migrations are independent. In the next section we will examine how dependent credit migrations can be introduced into the simulation process.

<table>
<thead>
<tr>
<th>Row</th>
<th>Duration time $t$ (in months)</th>
<th>Initial rating $i$</th>
<th>End rating $j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>36</td>
<td>AA</td>
<td>AA</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>AA</td>
<td>AAA</td>
</tr>
<tr>
<td>3</td>
<td>71</td>
<td>AA</td>
<td>AA</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>AA</td>
<td>BBB</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 1
Parts of the duration time table for rating class AA
3 Dependent Credit Migrations

To model dependent migrations in a credit portfolio, there has to be some approach for including the dependence in migrations. This can either be done by using correlations or a copula framework. We will start with an approach using correlations that has been initially suggested by Belkin et al. (1998b) and Kim (1999). Hereby, credit migrations are modeled as being dependent on a systematic risk factor and an idiosyncratic, firm-specific factor. The dependence in credit migrations can then be triggered by the degree of correlation with the systematic risk factor. We also suggest the use of copulas for modeling the joint dynamics of credit rating changes.

3.1 Dependence based on a Credit Cycle Index

We will first consider an approach based on a factor model including a systematic and idiosyncratic risk component as it has been suggested in Belkin et al. (1998b); Finger (2001); Kim (1999). The model was initially designed to adjust migration matrices for business cycle effects and derive conditional transition matrices instead of using average historical ones. In the first step, a so-called credit cycle index $Z_t$ is determined, that defines the credit state based on macroeconomic conditions shared by all obligors during period $t$. The index is designed to be positive in good days and to be negative in bad days. A positive index implies a lower PD and downgrading probability but a higher upgrading probability and vice versa. A forecast of the index $Z_t$ is then calculated based on the outcome of some macroeconomic variables.

Different degrees of dependence in migration behavior can then be added by adjusting the transition matrix according to an estimated or forecasted value of the credit cycle index. Hereby, it is assumed that rating transitions reflect an underlying continuous credit-change indicator $Y_t$ following a standard normal distribution. Further, the credit-change indicator is assumed to be influenced by both a systematic and unsystematic risk component. Therefore, $Y_t$ has a linear relationship with the systematic credit cycle index $Z_t$ and an idiosyncratic error term $u_t$. The typical one-factor model parametrisation (Belkin et al., 1998a; Finger, 2001) can then be denoted by:

$$Y_t = wZ_t + \sqrt{1-w^2}u_t.$$  \hfill (7)

Since both $Z_t$ and $u_t$ are scaled to the standard normal distribution, with the weights chosen to be $w$ and $\sqrt{1-w^2}$, $Y_t$ is also standard normally distrib-
uted. Hereby, $w^2$ provides a straightforward interpretation as the correlation between the the systematic credit cycle index $Z_t$ and the credit change indicator $Y_t$. The probability distribution for the rating change of a company then takes place according to the outcome of the systematic risk index. In particular, default happens when the value of $Y_t$ drops below a defined threshold $T$:

$$P(Y_t < T) = P(wZ_t + \sqrt{1 - w^2}u_t < T) =$$

$$= P(u_t < \frac{T - wZ_t}{\sqrt{1 - w^2}}) = \Phi\left(\frac{T - wZ_t}{\sqrt{1 - w^2}}\right)$$

Then the shift of the credit-change indicator $Y_t$ is dependent on the outcome of the credit cycle index $Z_t$. To extend this scheme to a multi-rating system, it is assumed that conditional on an initial credit rating $i$ at the beginning of a year, one partitions values of the credit change indicator $Y_t$ into a set of disjoint bins. The bins are defined in a way that the probability of $Y_t$ falling in a given interval equals the corresponding historical average transition rate. This can be done by simply inverting the cumulative normal distribution function starting from the default column what is illustrated in figure 1. The bins for the credit migrations can then be calculated by partitioning $(-\infty, \infty)$ into $K$ sub-intervals for each rating class $i$:

$$t_1 = \Phi^{-1}(1 - p_{i,1})$$

$$t_2 = \Phi^{-1}(1 - (p_{i,1} + p_{i,2}))$$

$$\ldots \quad \ldots$$

$$t_j = \Phi^{-1}(1 - \sum_{k=1}^{j} p_{i,k})$$

$$\ldots \quad \ldots$$

$$t_{K-1} = \Phi^{-1}(1 - \sum_{k=1}^{K-1} p_{i,k})$$

The migrations can then be described by the following function $f : [0, 1] \to S$: 

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Using the bins calculated from the average historical transition matrix, the conditional transition probabilities based on the outcome of the credit cycle index $Z_t$ can be calculated. On average days, one obtains $Z_t = 0$ for the systematic risk index. A positive outcome of the credit cycle index $Z_t$ shifts the credit-change indicator to the right-hand side while in the case of a bad outcome of the systematic risk index the distribution moves to the left-hand side. Thus, for each year with a positive or negative outcome of the systematic credit cycle index, the conditional transition rates will deviate from the average historical migration matrix.

### 3.2 Dependence based on individual transitions using copulas

Alternatively to the changes in migration behavior due to a credit cycle index, one can also add dependence between migration by using correlated random numbers $u_t$ for the simulation. Assume that a conditional or unconditional migration matrix has been calculated according to the predicted outcome of the credit cycle index $Z_t$. Then, using the thresholds (9) and the function (10), it is straightforward to simulate individual migrations for a loan or bond portfolio. In the case of the $u_i$ being uncorrelated, we can generate them by drawing
i.i.d. random numbers \((u_1, u_2, \ldots, u_n)\) from a standard normal distribution function \(\Phi\). However, using the Cholesky decomposition, it is straightforward to also include a dependence structure to the migrations. Let us therefore, denote the correlation matrix for the \(u_i\) by \(C\). Then \(C\) is a positive definite matrix and we can use the Cholesky decomposition to obtain matrices \(A\) and \(A^T\) such that
\[
C = A^T A
\]

The matrix \(A\) can then be used to transform a vector of uncorrelated random variates \((u_1, u_2, \ldots, u_n)\) into correlated correlated numbers \((u_1^c, u_2^c, \ldots, u_n^c)\) by:
\[
(u_1^c, u_2^c, \ldots, u_n^c) = (u_1, u_2, \ldots, u_n) \cdot A.
\]

To simulate correlated credit migrations, then simply function (10) is applied to determine the outcome of the next rating state.

Often a single statistical parameter like the linear correlation coefficient will not be able to capture the entire dependence structure between two random variables in the general case. In this case a general concept of describing the dependence structure within multivariate distributions is needed. Since marginal distributions are very illustrative, easy to handle and often used as basic building blocks for the design of a multivariate distribution, the idea of separating the description of the joint multivariate distribution into the marginal behaviour and the dependence structure is very attractive also for dependent migrations. Then the dependence between the individual marginal distributions is modeled by a copula. As mentioned above, the literature provides many suggestions of copulae for credit risk (Frey and McNeil, 2003; Giesecke, 2004; Laurent and Gregory, 2005; Hull and White, 2004; Li, 2000; Mashal and Naldi, 2002; Schönbucher and Schubert, 2001; Schönbucher, 2003) that will permit various dependence structures among the rating migrations. A copula is a function that combines the marginal distributions to form the joint multivariate distribution (Schweizer and Sklar, 1983):

**Definition 2** A copula is any real valued function \(C : [0, 1]^n \rightarrow [0, 1]\), i.e. a mapping of the unit hypercube into the unit interval, which has the following three properties:

1. \(C(u_1, \ldots, u_n)\) is increasing in each component of \(u_i\).
2. \(C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i\) for all \(i \in \{1, \ldots, n\}\), \(u_i \in [0, 1]\).
3. For all \((a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0, 1]^n\) with \(a_i \leq b_i\):
\[
\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1 + \cdots + i_n} C(u_{1i_1}, \ldots, u_{ni_n}) \geq 0
\]

where \( u_{j1} = a_j \) and \( u_{j2} = b_j \) for all \( j \in \{1, \ldots, n\} \).

Let \( X = (X_1, \ldots, X_n)' \) be a random vector of real-valued random variables whose dependence structure is completely described by the joint distribution function

\[
F(x_1, \ldots, x_n) = P(X_1 < x_1, \ldots, X_n < x_n). \tag{13}
\]

Each random variable \( X_i \) has a marginal distribution of \( F_i \) that is assumed to be continuous for simplicity. Recall that the transformation of a continuous random variable \( X \) with its own distribution function \( F \) results in a random variable \( F(X) \) which is standardly uniformly distributed. Thus transforming equation (13) component-wise yields

\[
F(x_1, \ldots, x_n) = P(X_1 < x_1, \ldots, X_n < x_n) \\
= P[F_1(X_1) < F_1(x_1), \ldots, F_n(X_n) < F_n(x_n)] \\
= C(F_1(x_1), \ldots, F_n(x_n)), \tag{14}
\]

where the function \( C \) can be identified as a joint distribution function with standard uniform marginals — the copula of the random vector \( X \). In equation (14), it can be clearly seen, how the copula combines the marginals to the joint distribution.

In the following we will restrict ourselves to the use of the Gaussian copula and the Student t-copula to simulate dependent credit migrations. However, the extension of the approach to using various other copulas is straightforward. The multivariate Gaussian copula as probably the simplest example:

\[
C^{\text{Normal}}(u_1, \ldots, u_n) = \Phi^0_{\Sigma}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)) \tag{15}
\]

Hereby, \( \Phi \) denotes the the standard normal cumulative distribution function, \( \Phi^{-1} \) the inverse of the standard normal cumulative distribution function and \( \Phi^0_{\Sigma} \) the standard multivariate Normal distribution with correlation matrix \( \Sigma \). As it was mentioned above, the multivariate normal copula correlates the random variables rather near the mean and not in the tails and, therefore, fails to incorporate tail dependence. To also add more dependence in the tails, alternatively, we also provide the framework using the Student t-copula. To generate correlated uniform numbers the Cholesky decomposition described above can be used, see e.g. Cherubini et al. (2004). The algorithm can be summarized as follows:
(1) Find the Cholesky decomposition of $R = A^T A$.
(2) Simulate $n$ independent random numbers $(u_1, \ldots, u_n)$ from $N(0,1)$.
(3) Set 
$$ (x_1, x_2, \ldots, x_n) = A^T \cdot (u_1, \ldots, u_n)' .$$
(4) Set $(u_1^R, u_2^R, \ldots, u_n^R) = (\Phi(x_1), \Phi(x_2), \ldots, \Phi(x_n))$ where $\Phi$ denotes distribution function of univariate standard normal distribution.

On the other hand, to generate random numbers from the Student $t$-copula with $v$ degrees of freedom, the following algorithm can be used:

(1) Find the Cholesky decomposition of $R = A^T A$.
(2) Simulate $n$ independent random numbers $(u_1, \ldots, u_n)$ from $N(0,1)$.
(3) Simulate a random number $s$ from $\chi^2_v$ independent of $z$.
(4) Set $(y_1, y_2, \ldots, y_n) = A^T \cdot (u_1, \ldots, u_n)'$.
(5) Set $(x_1, x_2, \ldots, x_n) = \sqrt{(v/s)} \cdot (y_1, y_2, \ldots, y_n)$.
(6) Set $u_i = T_v(x_i)$ for $i = 1, \ldots, n$ where $T_v$ denotes the univariate Student $t$ distribution function.

Similar algorithms are available to simulate random numbers from other copulas like Gumbel, Clayton or Frank copula, see e.g. Cherubini et al. (2004). While the Gaussian and Student $t$-copula are symmetric alternative copulas will also allow for more complex dependence structures for rating migrations. As the results in the next section indicate, not only the estimated degree of dependence, but also the choice of the copula significantly affects the results on credit migrations.

4 An Empirical Study on Dependent Migrations

4.1 Conditional Migrations

In a first step we simulate conditional migration matrices based on the one-factor model suggested in Belkin et al. (1998b); Kim (1999) and compare to actually observed transition matrices. Hereby, we use the average migration matrix $\bar{P}$ for the years 1982-2001 that is displayed in table and the empirically observed yearly transition matrices for this period. Following the suggested model, in a first step we draw random numbers for the systematic risk factor $Z \sim N(0,1)$. Based on the outcome of the systematic risk factor and the factor $w$ measuring the sensitivity of the individual credit cycle index to the systematic risk factor, a conditional migration matrix is derived. The literature reports different results for the appropriate weight of the factor $w$. While Kim
(1999) estimates \( w = 0.05 \) for investment grade and \( w = 0.3384 \) for speculative grade bonds, later studies (Wei, 2003; Trück, 2008) find \( w \) between 0.05 – 0.17 for investment grades and between 0.16 – 0.23 for speculative grade bonds. Various values for \( w \) were tried in order to determine how the sensitivity factor affects deviations from the average transition matrix that are in line with empirically observed yearly migration matrices. The distance from the average migration matrix \( \bar{P} \) was measured using the mobility metric by Jafry and Schuermann (2004) and the weighted index to default by Trück and Rachev (2008). While the former concentrates on the mobility inherent in a transition matrix, the second one puts more focus on comparing transition matrices with respect to the risk of downgrades or defaults for a credit portfolio.

<table>
<thead>
<tr>
<th></th>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.9276</td>
<td>0.0661</td>
<td>0.0050</td>
<td>0.0009</td>
<td>0.0003</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
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<td>0.0064</td>
<td>0.9152</td>
<td>0.0700</td>
<td>0.0062</td>
<td>0.0008</td>
<td>0.0111</td>
<td>0.0002</td>
<td>0.0001</td>
</tr>
<tr>
<td>A</td>
<td>0.0007</td>
<td>0.0221</td>
<td>0.9137</td>
<td>0.0546</td>
<td>0.0058</td>
<td>0.0024</td>
<td>0.0003</td>
<td>0.0005</td>
</tr>
<tr>
<td>Baa</td>
<td>0.0005</td>
<td>0.0029</td>
<td>0.0550</td>
<td>0.8753</td>
<td>0.0506</td>
<td>0.0108</td>
<td>0.0021</td>
<td>0.0029</td>
</tr>
<tr>
<td>Ba</td>
<td>0.0002</td>
<td>0.0011</td>
<td>0.0052</td>
<td>0.0712</td>
<td>0.8229</td>
<td>0.0741</td>
<td>0.0111</td>
<td>0.0141</td>
</tr>
<tr>
<td>B</td>
<td>0.0000</td>
<td>0.0010</td>
<td>0.0035</td>
<td>0.0047</td>
<td>0.0588</td>
<td>0.8323</td>
<td>0.0385</td>
<td>0.0612</td>
</tr>
<tr>
<td>C</td>
<td>0.0012</td>
<td>0.0000</td>
<td>0.0029</td>
<td>0.0053</td>
<td>0.0157</td>
<td>0.1121</td>
<td>0.6238</td>
<td>0.2389</td>
</tr>
<tr>
<td>D</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 2

To determine the Jafry-Schuermann mobility metric, first a so-called mobility matrix \( \tilde{P} \) is calculated by subtraction of the identity matrix \( I \) from the original transition matrix \( P \). The metric is then defined as the average of the singular values of the mobility matrix:

\[
M_{SVD}(P) = \frac{\sum_{i=1}^{n} \sqrt{\lambda_i(\tilde{P}^n\tilde{P})}}{n}.
\]  

(16)

The authors show that this metric captures the so-called ‘amplification factor’ or the dynamic part of the migration matrix. Therefore, it approximates the average probability of migration which can be considered as a meaningful magnitude calibration for a metric. To measure the difference between two migration matrices in terms of mobility, one can then calculate:

\[
D_{SVD}(P, Q) = M_{SVD}(P) - M_{SVD}(Q).
\]  

(17)
Fig. 2. Distribution of difference from average transition matrix $\hat{P}$ for simulated conditional transition matrices with $w = 0.20$ (left panel) and empirically observed migration matrices 1982-2001 (right panel) according to WID.

The expression (17) gives a directional deviation between two matrices in terms of the mobility or approximate average probability of migration. (Trück and Rachev, 2008) derive several criteria to derive risk-sensitive difference indices for migration matrices. Among others, they suggest the so-called weighted index to default (WID) as an appropriate measure for the comparison of transition matrices. For this index, the weights for the individual cells are chosen the following way:

$$WID(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n-1} d(i, j) + \sum_{i=1}^{n} n^2 \cdot d(i, n)$$

As pointed out by the authors the index concentrates mainly on the riskiness of a transition matrix for a credit portfolio. Figure 2 shows the difference of the simulated conditional migration matrices from the average one as well as the actually observed distance of the empirically observed ones $WID(P, \bar{P})$.

The results in Figure 2 indicate that for the values suggested in the literature, the differences between average historical and empirically observed yearly migration matrices are larger than suggested by the models. Similar results are also obtained using the Jafry-Schuermann mobility metric. Unless the factor $w$ is chosen substantially higher than the values suggested in the literature, the empirically observed extreme changes in migration behavior cannot be captured. These changes are usually due to substantially higher or lower defaults, downgrade and upgrade rates, or mobility than suggested by the average historical migration matrix. For lower values of $w$, migration matrices as they could be observed e.g. in 1996 or 2001 have very low probability $p < 0.01$ according to conditional model. Therefore, it seems to be reasonable to introduce additional dependence based on individual migrations using a copula.
structure into the model.

4.2 Including Dependence using Copulas

To illustrate the usefulness of the suggested approach in this section we present empirical results on simulating dependent migrations using the Gaussian and Student t-copula. We compare the different simulation methods in the following way: using an exemplary portfolio and Moody’s average historical migration matrix from 1982-2001 in table 2, we apply the Gaussian and Student t-copula to simulate dependent credit migrations for a one-year time horizon and compare these results to independently simulated transitions. Our exemplary loan portfolio consists of 1120 companies with rating distribution and exposures according to table 4.2. After choosing a copula correlation parameter \( \rho \) for the Gaussian and Student t-copula, for each scenario \( n = 1000 \) simulations for the portfolio were run. In a first step we investigate the distribution of the number of defaults in the portfolio dependent on the chosen copula correlation parameter. Hereby, we illustrate the case of independent migrations as well as dependent migrations from the Gaussian and t-copula model for different choices of the copula correlation parameter \( \rho \). For our study we decided to compare three different cases for the correlation parameter: \( \rho = 0.1 \), \( \rho = 0.2 \) and finally \( \rho = 0.5 \). Note that the latter will be rather unrealistic in empirical applications, however, to illustrate the significant effect, dependencies in migration behavior may have, we decided to also allow for such a comparably high value of the copula correlation parameter.

<table>
<thead>
<tr>
<th>Rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.</td>
<td>11</td>
<td>106</td>
<td>260</td>
<td>299</td>
<td>241</td>
<td>95</td>
<td>148</td>
</tr>
<tr>
<td>Average Exposure (Mio. Euro)</td>
<td>20</td>
<td>15</td>
<td>15</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 3
Ratings and exposures for the considered credit portfolio.

Figure 3 provides the number of defaults in the portfolio after one year for independent migrations, dependent migrations from the Gaussian and Student t-copula for the different copula correlation parameters \( \rho = 0.1 \), \( \rho = 0.2 \) and \( \rho = 0.5 \). The corresponding descriptive statistics are given in Table 4. Obviously we find that the simulated distribution for the number of defaults is almost symmetric in the case of independence. We observe a mean of approximately 46 defaults and a 90%-confidence interval for the number of defaults would be \([35, 56]\). What we observe from figure 3 is that the higher the copula correlation parameter, the more right-skewed becomes the distribution of defaults and the higher becomes the standard deviation of the distribution. While for the case of independent migration the simulated distribution has a standard deviation of \( \sigma = 6.24 \), for an increase in the copula correlation para-
Fig. 3. Number of defaults in the portfolio after one year for independent migrations (upper left panel), for dependent migrations from Gaussian copula $\rho = 0.1$ (upper right panel), $\rho = 0.2$ (mid left panel) and $\rho = 0.5$ (mid right panel) and for dependent migrations using the Student t copula with $\rho = 0.2$ (lower left panel and $\rho = 0.5$ (lower right panel).

The corresponding numbers are 16.05 (for $\rho = 0.1$), 23.37 (for $\rho = 0.2$), and 38.29 (for $\rho = 0.5$). This also has a dramatic effect on the confidence intervals for the number of defaults: for $\rho = 0.5$ the 90%-confidence interval for the number of defaults is [3, 121].
Table 4
Descriptive statistics for the simulated distribution of the number of defaults in the portfolio after one year for independent migrations and dependent migrations from Gaussian and Student t copula.

<table>
<thead>
<tr>
<th>Dependence</th>
<th>$q_{0.01}$</th>
<th>$q_{0.05}$</th>
<th>Mean</th>
<th>$q_{0.95}$</th>
<th>$q_{0.99}$</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 0$</td>
<td>31</td>
<td>35</td>
<td>45.6100</td>
<td>56</td>
<td>59</td>
<td>6.2436</td>
</tr>
<tr>
<td>Gaussian $\rho = 0.1$</td>
<td>15</td>
<td>22</td>
<td>45.1000</td>
<td>74</td>
<td>90</td>
<td>16.0518</td>
</tr>
<tr>
<td>Gaussian $\rho = 0.2$</td>
<td>9</td>
<td>14</td>
<td>46.0860</td>
<td>90</td>
<td>118</td>
<td>23.3681</td>
</tr>
<tr>
<td>Gaussian $\rho = 0.5$</td>
<td>0</td>
<td>3</td>
<td>45.4780</td>
<td>121</td>
<td>143</td>
<td>38.2871</td>
</tr>
<tr>
<td>Student t $\rho = 0.1$</td>
<td>6</td>
<td>11</td>
<td>45.9920</td>
<td>92</td>
<td>127</td>
<td>25.2758</td>
</tr>
<tr>
<td>Student t $\rho = 0.2$</td>
<td>1</td>
<td>7</td>
<td>46.9270</td>
<td>104</td>
<td>139</td>
<td>31.5953</td>
</tr>
<tr>
<td>Student t $\rho = 0.5$</td>
<td>0</td>
<td>1</td>
<td>46.0300</td>
<td>125</td>
<td>157</td>
<td>42.0739</td>
</tr>
</tbody>
</table>

The effects are even more pronounced when the Student t-copula is used adding more dependence in the tails. Choosing $\rho = 0.5$, for a small number of simulations even more than 200 defaults could be observed for the portfolio. In general, the distribution of defaults is even more skewed and exhibits higher variance than for the Gaussian copula with the same coefficient of correlation. Note, however, that despite the substantial changes in the shape and variance of the distribution the average number of defaults for all simulation experiments is around 46. So the mean number of defaults is not affected by the copula or correlation parameter. On the other hand, for the distribution and calculated default quantiles, both the choice of the copula and the correlation parameter yields significantly different results. Still, these results are not surprising and could also be obtained by using copulas just to model dependent defaults only. Therefore, we will now also have a look at the distribution of ratings for the exemplary portfolio what can usually not be investigated by modeling dependent defaults.

4.3 The Distribution of Rating Changes

Let us first have a look at the non-investment grade rating class $C$. Recall that for the usual portfolio there were approximately 150 companies in this rating state. Figure 4 displays the number of companies in the rating state after one year. Again the simulation results are provided for independent migrations and for dependent migrations from the Gaussian copula and Student t-copula with different correlation parameters. In the case of independence, the simulated distribution for the number of companies in rating state $C$ is quiet symmetric. Again we find for the Gaussian copula, that the higher the
Table 5
Descriptive statistics for the simulated distribution of the number of companies in rating state 'C' after one year for independent migrations and dependent migrations from Gaussian and Student t copula.

<table>
<thead>
<tr>
<th>Dependence</th>
<th>$q_{0.01}$</th>
<th>$q_{0.05}$</th>
<th>Mean</th>
<th>$q_{0.95}$</th>
<th>$q_{0.99}$</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 0$</td>
<td>85</td>
<td>89</td>
<td>99.2760</td>
<td>110</td>
<td>113</td>
<td>6.4041</td>
</tr>
<tr>
<td>Gaussian $\rho = 0.1$</td>
<td>73</td>
<td>84</td>
<td>99.8740</td>
<td>114</td>
<td>120</td>
<td>8.9141</td>
</tr>
<tr>
<td>Gaussian $\rho = 0.2$</td>
<td>52</td>
<td>72</td>
<td>98.8960</td>
<td>117</td>
<td>125</td>
<td>14.0366</td>
</tr>
<tr>
<td>Gaussian $\rho = 0.5$</td>
<td>17</td>
<td>38</td>
<td>98.8940</td>
<td>134</td>
<td>152</td>
<td>29.5456</td>
</tr>
<tr>
<td>Student t $\rho = 0.1$</td>
<td>28</td>
<td>47</td>
<td>99.3500</td>
<td>144</td>
<td>156</td>
<td>29.7486</td>
</tr>
<tr>
<td>Student t $\rho = 0.2$</td>
<td>29</td>
<td>45</td>
<td>98.3220</td>
<td>144</td>
<td>157</td>
<td>30.2874</td>
</tr>
<tr>
<td>Student t $\rho = 0.5$</td>
<td>12</td>
<td>30</td>
<td>97.9860</td>
<td>150</td>
<td>165</td>
<td>37.5379</td>
</tr>
</tbody>
</table>

choice of the correlation parameter, the more skewed and volatile becomes the distribution. Note, however, that in this case the distribution is skewed to the left and not to the right. Again, for the Student t-distribution the effects on the volatility of the rating distribution are more pronounced. The standard deviation of the number of companies in rating class C ranges from approximately 29.75 to 37.54 depending on the choice of the copula correlation parameter. In comparison, the standard deviation for independently simulated transitions is 6.40. Despite the higher volatility, for the Student t-copula, the simulated distributions of companies in rating class C are less skewed than for the Gaussian copula. Overall, we find that when applying a model with independent individual migrations the number of companies in rating state C would be expected to be reduced from 148 to somewhere between 89 and 110 with 90% confidence. Assuming that credit migrations show a dependence structure that could be modeled by a Gaussian or Student t-copula, this interval becomes substantially wider. The simulated 90% confidence interval for example is [72, 117] for a Gaussian copula with correlation parameter $\rho = 0.2$, while it increases to [45, 144] when a Student t-copula with the same copula correlation parameter is used. This means that while the expected value of companies in a rating state stays approximately the same, confidence intervals significantly become wider when the dependence for credit migrations is increased or when a copula with more dependence in the tails is used.

We finally investigate the number of companies in the investment grade rating state A after one year. Initially, there were 260 companies in this rating state. Figure 5 displays the number of companies in rating state A after one year. Also here the simulation results are provided for independent migrations, for dependent migrations from Gaussian copula and Student t-copula with the

21
Companies in 'A'

<table>
<thead>
<tr>
<th>Dependence</th>
<th>$q_{0.01}$</th>
<th>$q_{0.05}$</th>
<th>Mean</th>
<th>$q_{0.95}$</th>
<th>$q_{0.99}$</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 0$</td>
<td>249</td>
<td>253</td>
<td>263.5820</td>
<td>275</td>
<td>278</td>
<td>6.7798</td>
</tr>
<tr>
<td>Gaussian $\rho = 0.1$</td>
<td>231</td>
<td>241</td>
<td>263.7470</td>
<td>289</td>
<td>303</td>
<td>14.6565</td>
</tr>
<tr>
<td>Gaussian $\rho = 0.2$</td>
<td>191</td>
<td>227</td>
<td>262.5520</td>
<td>300</td>
<td>331</td>
<td>23.4282</td>
</tr>
<tr>
<td>Gaussian $\rho = 0.5$</td>
<td>131</td>
<td>197</td>
<td>265.0690</td>
<td>339</td>
<td>411</td>
<td>45.3503</td>
</tr>
<tr>
<td>Student t $\rho = 0.1$</td>
<td>101</td>
<td>178</td>
<td>263.2730</td>
<td>330</td>
<td>364</td>
<td>44.5682</td>
</tr>
<tr>
<td>Student t $\rho = 0.2$</td>
<td>116</td>
<td>180</td>
<td>265.0640</td>
<td>333</td>
<td>390</td>
<td>46.8939</td>
</tr>
<tr>
<td>Student t $\rho = 0.5$</td>
<td>73</td>
<td>146</td>
<td>263.4040</td>
<td>370</td>
<td>443</td>
<td>60.9158</td>
</tr>
</tbody>
</table>

Table 6
Descriptive statistics for the simulated distribution of the number of companies in rating state 'A' after one year for independent migrations and dependent migrations from Gaussian and Student t copula.

same correlation parameters $\rho = 0.1$, $\rho = 0.2$ and $\rho = 0.5$. For all simulation experiments the expected value of companies in the rating state after one year is slightly increased to a number between 262 and 265. However, while the mean of the distribution remains approximately the same, the shape of the distribution is substantially affected by the chosen copula and the correlation parameter. For the Gaussian copula, we find, that the higher the choice of $\rho$, the more leptokurtic the distribution becomes. Further the standard deviation of the distribution increases substantially from $\sigma = 6.78$ in case of independent migrations to $\sigma = 45.35$ when the Gaussian copula with $\rho = 0.5$ is applied. However, unlike for the number of defaults or companies in rating state C, we find that the distribution remains rather symmetric in this case. As expected, for the Student t-copula the distribution is more leptokurtic and exhibits even higher standard deviation. Simulated 90% confidence intervals now range from $[253, 275]$ for independence to $[146, 370]$ when a Student t copula with copula correlation parameter $\rho = 0.5$ is used. Note that this interval is approximately ten times wider than in case of independence.

Overall, the results point out the substantial effects of dependent credit migrations. Hereby, unlike for models that concentrate on dependent defaults only, a simulation model for dependent credit migrations is able to provide results on the number of companies that are expected to be in a certain rating class after an arbitrary number of periods. In our analysis only results for $t = 1$ year were provided, however it is straightforward to extend the time horizon. We find that while both for the number of defaults and number of companies in a rating state, the expected value stays approximately the same, simulated confidence intervals are significantly wider when the copula correlation parameter is increased or a copula with more tail dependence is used.
Fig. 4. Number of companies in rating state 'C' after one year for independent migrations (upper left panel), for dependent migrations from Gaussian copula $\rho = 0.1$ (upper right panel), $\rho = 0.2$ (mid left panel) and $\rho = 0.5$ (mid right panel) and for dependent migrations using the Student t copula with $\rho = 0.2$ (lower left panel) and $\rho = 0.5$ (lower right panel).

Further the shape of the distribution changes significantly and becomes either more skewed or leptokurtic, dependent on the rating state. This will have substantial effects not only for the VaR of a credit portfolio, but also when rating changes are considered for a mark-to-market evaluation of the portfolio.
Fig. 5. Number of companies in rating state ‘A’ after one year for independent migrations (upper left panel), for dependent migrations from Gaussian copula $\rho = 0.1$ (upper right panel), $\rho = 0.2$ (mid left panel and $\rho = 0.5$ (mid right panel) and for dependent migrations using the Student t copula with $\rho = 0.2$ (lower left panel and $\rho = 0.5$ (lower right panel).
5 Conclusion

Considering the significant and sudden changes in creditworthiness of many obligors during the financial crisis, scenario analysis techniques as well as models quantifying possible extreme outcomes in rating changes for a portfolio are required. In this paper we reviewed different approaches for simulation of credit migrations and provided techniques that focus on modeling dependencies in rating transitions within a factor model and copula framework. Copulas have become a very popular tool for modeling dependencies for default-risky instruments such that quite an amount of literature has been dedicated to the topic in recent years (Frey and McNeil, 2003; Giesecke, 2004; Laurent and Gregory, 2005; Hull and White, 2004; Li, 2000; Mashal and Naldi, 2002; McNeil et al., 2005; Schönbucher and Schubert, 2001; Schönbucher, 2003). On the other hand there are only a limited number of publications focusing on dependencies also in credit migrations, exceptions include Hamilton et al. (2001); Gagliardini and Gourieroux (2005a,b); McNeil and Wendin (2006). We extend the literature on the issue by applying an approach modeling credit migrations as being dependent on a systematic risk factor in combination with a correlation or copula framework. In this approach, the dependence in credit migrations can be triggered by the degree of correlation with a systematic risk factor, in combination with the use of copulas for modeling the joint dynamics of credit rating changes. We apply the Gaussian and Student t-copula to show the effects of different assumptions about the degree of dependence and using different copulas. The application of the model using alternative asymmetric copulas like the Gumbel, Clayton or Frank copula is straightforward.

Our results indicate that both the integration of dependence and the choice of the copula have substantial impacts on Value-at-Risk, the loss distribution and also the distribution of the number of companies in the individual rating classes. We further find that while generally the mean of these distribution remains unchanged with more dependence in particular the risk in the tails of the rating and default distributions increases, giving a higher probability to extreme outcomes. If the dependence parameter for joint credit migrations is increased up to a certain level not only a significant increase in default probabilities, but also extreme scenarios including the downgrade of a substantial fraction of loans can be generated. We conclude that this simple but appropriate framework for simulating dependent credit migrations can be considered as particular helpful for risk managers in the financial industry. The use of dependent credit migrations will provide financial institutions with the possibility to investigate the effects of correlations on the distribution of ratings what might turn out to be of high impact in particular in the long run. In particular, the model will be helpful to investigate extreme correlation and dependence scenarios as they could recently be observed during the financial crises. Therefore, our approach provides a helpful tool for stress testing in
credit portfolio analysis. Future work includes the further investigation on the choice of copulas for dependent credit migrations and appropriate estimation techniques for copula and correlation parameters within this model.
References