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Optimal VWAP Trading and Relative Volume

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Abstract

Volume Weighted Average Price (VWAP) for a traded financial asset is total traded value divided by total traded volume. It is a quality of execution metric popular with institutional traders for measuring the price impact of trading. VWAP is also a ‘virtuous trade’ that minimizes price impact by spreading the liquidity demand of large orders across the trading period. The optimal mean-variance VWAP trading strategy is derived in this paper, by projecting from a space of strategies defined by a filtration enlarged by knowledge of total traded volume to the space of strategies defined by the observed (trader accessible) filtration. The optimal mean-variance VWAP trading strategy is linear combination of a variance-minimizing VWAP strategy and a ‘price directional’ trading strategy, which are obtained as explicit closed-form solutions.

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1 Introduction

A general mean-variance optimal VWAP trading strategy is formulated. A minimum mean-square ($L^2$) strategy is developed and then extended to a general optimal strategy using the Markowitz [13] quadratic mean-variance utility function.

Volume Weighted Average Price (VWAP) trading is used by large (institutional) traders to trade large orders in financial markets. Implicit in the use of VWAP trading is the recognition that large orders traded in financial markets may trade at an inferior price compared to smaller orders. This is known as the liquidity impact cost or market impact cost of trading large orders.

The VWAP price as a quality of execution measurement was first developed by Berkowitz, Logue and Noser [1]. They argue (page 99) that ‘a market impact measurement system requires a benchmark price that is an unbiased estimate of prices that could be achieved in any relevant trading period by any randomly selected trader’ and then define VWAP as an appropriate benchmark that satisfies this criteria. An optimal VWAP trading strategy has been formulated by Konishi [11] for price modelled as a Wiener process with no drift in a formal market where trade size is restricted to 1 stock unit.

VWAP is naturally defined using relative volume $Y$ rather than cumulative volume $V$. The relative volume process $Y$ is defined as market intra-day cumulative volume $V_t$ divided by market total final volume $V_T$ and is adapted to a filtration $\mathcal{G}$ which is the accessible (observed) filtration $\mathcal{F}$ enlarged by knowledge of final volume $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(V_T)$. The VWAP price $V_T$ (eqn 1) at time $T$ can be formulated as an Itô integral under $\mathcal{G}$ with a continuous price $X$ integrator and with a $\mathcal{G}$-predictable relative volume process integrand $\xi_t^{\mathcal{G}} = 1 - Y_t$. Thus no $\mathcal{F}$ attainable (Schweizer [22]) VWAP trading strategy exists.

$$\mathcal{V}_T = X_0 + \int_0^T \xi_t^{\mathcal{G}} \, dX$$  \hspace{1cm} (1)

The $\mathcal{F}$ adapted mean square optimal (minimal $L^2$) trading strategy $\xi_t^{\mathcal{F}}$ (eqn 2) that minimizes the square of the difference between traded VWAP and market VWAP is explicitly derived. This is an optimal strategy under restricted information examined by Schweizer [21] where $X$ is martingale and generalized by Møller [15] [16] to semimartingale $X$. 

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\[ \xi^{V,\mathcal{F}} = \min_{\xi^F} \mathbb{E} \left[ \left( \int_0^T (\xi^{V,\mathcal{F}} - \xi^F) \, dX \right)^2 \right] \] (2)

The mean square optimal strategy then extended by explicitly deriving the \( \mathcal{F} \) adapted Markowitz mean-variance optimal trading strategy \( \xi^{\lambda,\mathcal{F}} \) (eqn 3):

\[ \xi^{\lambda,\mathcal{F}} = \max_{\xi^F} \left[ \mathbb{E} \left[ \int_0^T (\xi^{V,\mathcal{F}} - \xi^F) \, dX \right] - \frac{1}{2\lambda} \text{Var} \left[ \int_0^T (\xi^{V,\mathcal{F}} - \xi^F) \, dX \right] \right] \] (3)

The optimal mean variance VWAP trading strategy \( \xi^{\lambda,\mathcal{F}} \) is conceptually two distinct trading strategies (McCulloch [14]), one is a minimum variance VWAP hedging strategy, the other a ‘price directional’ strategy independent of the hedging strategy. The ‘price directional’ strategy uses the properties of the Variance Optimal Martingale Measure (VOMM) for additional variance and expected return.
2 Modelling VWAP

2.1 Price and Volume

Assumption 2.1.

(i) The stochastic environment of the VWAP asset is endowed with a stochastic basis $(\Omega, \mathcal{F}_T, \mathcal{F}, \mathbb{P})$ with a standard (right continuous and complete) filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$. The filtration $\mathcal{F}$ contains all accessible information about the VWAP asset. In particular, the price process $X$ is assumed to be an $\mathcal{F}$ continuous semimartingale and the cumulative volume process $V$ an $\mathcal{F}$ semimartingale.

(ii) VWAP trading is normally conducted on single financial market assets. Therefore the exposition will assume univariate price $X$ and cumulative volume $V$ processes.

(iii) For simplicity and clarity, it is assumed that the proportion $\rho$ of total final volume $V_T$ traded by the VWAP trader is not significant, $\rho << 1$.

The relative volume process $Y$ is the integer valued cumulative volume process divided by the final cumulative volume at the end of the VWAP period $Y_t = V_t/V_T$. The process is zero at time zero $Y_0 = 0$, monotonically non-decreasing and has a value of 1 at end of the VWAP period $Y_T = 1$. It is measurable with respect to a filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(V_T)$ initially enlarged by knowledge of the $\mathcal{F}_T \equiv \mathcal{G}_T$ measurable final cumulative volume $V_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ (abbreviated $L^2(\mathbb{P})$) and is defined on the stochastic basis $(\Omega, \mathcal{F}_T, \mathcal{G}, \mathbb{P})$ where $\mathcal{G} = \{\mathcal{G}_t\}_{0 \leq t \leq T}$ is a standard filtration.

Assumption 2.2. The relative volume process $Y$ is a monotonically increasing discrete square integrable $\mathcal{G}$ semimartingale with a finite number of values in the interval $[0, 1]$, $Y \in \mathcal{S}^2(\mathbb{P}, \mathcal{G})$.

Remark 2.3. Final cumulative volume $V_T \in \mathbb{Z}^+$ is a positive integer valued finite process, $V_T < \infty$. Therefore the natural filtration $\sigma(V_T)$ of $V_T$ is countably large and every $\mathcal{F}$ semimartingale is a $\mathcal{G}$ semimartingale (Jacod [8], Protter [18]). In particular, the $\mathcal{F}$ semimartingale price process $X$ is also a $\mathcal{G}$ semimartingale.
Assumption 2.4. The price process $X$ is a strictly positive continuous square integrable semimartingale with respect to the enlarged filtration $\mathcal{G}$, $X \in S^2(\mathbb{P}, \mathcal{G})$.

Figure 1: This graph shows typical relative volume trajectories for 3 stocks representing low, medium and high turnover stocks. The red line is the expected relative volume $\mathbb{E}[Y_t]$ for all stocks trading more than 50 trades a day on the NYSE. SUS is Storage USA, TXT is Textron Incorporated and TXN is Texas Instruments; on 2 Jul 2001 these stocks recorded 101, 946 and 2183 trades correspondingly.
2.2 A Continuous Time Model of VWAP

One the reasons for the popularity of VWAP as a measure of order execution quality is the simplicity of it’s definition - the total value of admissable\(^1\) trades divided by the total volume of all trades. If \(X_i\) and \(\Delta V_i\) are the price and volume respectively of the \(i\)th trade in the VWAP period \([0, T]\) with \(N_T\) total trades, then the VWAP price \(\mathcal{V}_T\) calculated at time \(T\) is readily computed as:

\[
\mathcal{V}_T = \frac{\text{total traded value}}{\text{total traded volume}} = \frac{\sum_{i=1}^{N_T} X_i \Delta V_i}{\sum_{i=1}^{N_T} \Delta V_i}
\]

Alternatively, the definition of VWAP can be written in continuous time notation. Let \(V_t\) be the cumulative volume traded at time \(t\) and \(X\) be the price process on a market that trades on the time interval \(t \in [0, T]\). The price process \(X\) is a \(\mathcal{G}\) semimartingale (remark 2.3), therefore VWAP can be formulated as a \(\mathcal{G}\) adapted Itô integral.

\[
\mathcal{V}_T = \frac{\text{total traded value}}{\text{total traded volume}} = \frac{1}{V_T} \int_0^T X_- dV
\]

(4)

Examining the integral above, it is intuitive that it relates to the relative volume process \(Y_t = V_t/V_T\). This is a key insight; VWAP is naturally defined using relative volume \(Y_t\) rather than actual volume \(V_t\).

\[
\mathcal{V}_T = \frac{1}{V_T} \int_0^T X_- dV = \int_0^T X_- dY
\]

Integration by parts gives:

\[
\mathcal{V}_T = X_T Y_T - X_0 Y_0 - \int_0^T Y_- dX - [X, Y]_T
\]

\(^1\)Not all trades are accepted as admissable in a VWAP calculation. Admissible trades are determined by market convention and are generally on-market trades. Off-market trades and crossings are generally excluded from the VWAP calculation because these trades are often priced away from the current market and represent volume in which a ‘randomly selected trader’ [1] cannot participate.
Since the relative volume $Y$ process is monotonically non-decreasing and therefore of finite variation and $X$ is $\mathcal{G}$ continuous by assumption 2.4, the quadratic covariation term is zero $[X, Y] = 0$. Also $Y_0 = 0$ and $Y_T = 1$ by definition, so the integration by parts equation simplifies to:

$$
\mathcal{V}_T = X_T - \int_0^T Y_- dX = X_0 + \int_0^T (1 - Y_-) dX
$$

Defining the $\mathcal{G}$ predictable integrand:

$$
\xi_t^{\mathcal{V}, \mathcal{G}} = 1 - Y_t^- = 1 - \frac{V_t^-}{V_T} \tag{5}
$$

Using this definition, VWAP has the following Itô integral representation.

$$
\mathcal{V}_T = X_0 + \int_0^T \xi_t^{\mathcal{V}, \mathcal{G}} dX
$$

**Remark 2.5.** Note that unlike a conventional trading strategy where the constant term represents initial trading capital, the $X_0$ term above (eqn 1) is initial stock price and cannot be specified by a VWAP trader. Consequently, this constant term plays no role in optimizing VWAP trading strategies.

Since the price process is square integrable by assumption and the relative volume process $0 \leq Y_t \leq 1$ is bounded, the VWAP random variable $\mathcal{V}_T$ is square integrable $\mathcal{V}_T \in L^2(\mathbb{P})$.

### 2.2.1 VWAP is Not Attainable in the Observed Filtration

**Definition** A random variable $U_T \in L^2(\mathbb{P})$ is $\mathcal{F}$ attainable (Schweizer [22]) if a unique $\mathcal{F}$ adapted predictable process $\gamma^{U, \mathcal{F}}$ exists such that $U_T$ is replicated by the following Itô integral.

$$
U_T = U_0 + \int_0^T \gamma^{U, \mathcal{F}} dX
$$
Lemma 2.6. VWAP is not $\mathcal{F}$ attainable.

Proof. By contradiction. Assume VWAP is $\mathcal{F}$ attainable and an attainable $\mathcal{F}$ predictable strategy $\xi^{V,F}$ exists, then from eqn 1 above:

$$V_T = \int_0^T \xi^{V,F} dX = \int_0^T (1 - Y_\cdot) dX$$

By the properties of the Itô integral (Jacod and Shiryaev [9]).

$$1_{[0,T]}(t) \xi^{V,F}_t dX_t = 1_{[0,T]}(t) (1 - Y_{t-}) dX_t$$

But $Y_t$ is not $\mathcal{F}_t$ measurable for $0 \leq t < T$ because, by definition, final cumulative volume $V_T$ not $\mathcal{F}_t$ measurable $0 \leq t < T$. Therefore no $\mathcal{F}$ attainable strategy $\xi^{V,F}$ exists.

$\square$
3 Optimal VWAP Trading Strategies

Since VWAP is not attainable in the accessible filtration $\mathcal{F}$ the objective is to specify a feasible risky optimal $\mathcal{F}$ adapted trading strategy. This is similar to the optimal quadratic projection of random variables $H \in L^2(\mathbb{P})$ onto a subspace of Itô integrals $(\xi \cdot X)_T \in L^2(\mathbb{P})$. The extensive quadratic hedging literature includes Chunli and Karatzas [7], Schweizer [22] [23], Delbaen and Schachermayer [3], Rheinländer and Schweizer [20], Pham and Rheinländer and Schweizer [17], Gouriéroux and Laurent and Pham [6] and Rheinländer [19].

Optimal VWAP trading belongs to the class of hedging problems where the accessible filtration $\mathcal{F}$ is smaller than the filtration $\mathcal{G}$ required for hedge replication to be attainable. This has been studied by Schweizer [21] for martingale $X$ and has also been studied by Föllmer and Schweizer [4] who gave a solution that was local error minimizing for semimartingale $X$. Møller [15] [16] and Schweizer [24] extended this to a Markowitz mean-variance optimal solution for semimartingale $X$.

This section formulates a minimum mean-square ($L^2$) VWAP strategy for martingale $X$, extends this to a mean-square ($L^2$) optimal strategy for semimartingale $X$ (eqn 2) and finally develops a Markowitz mean-variance optimal VWAP strategy ($\lambda \geq 0$) for semimartingale $X$ (eqn 3).

3.1 Preliminaries

3.1.1 The Existence of a $\mathcal{G}$ Equivalent Martingale Measure $\mathbb{Q}^{\mathcal{G}} \sim \mathbb{P}$

The definite Itô integral with integrand $\xi$ is notated:

$$G_T(\xi) = \int_0^T \xi \, dX$$

Let $L(X, \mathcal{G})$ denote the space of all $X$-integrable $\mathcal{G}$ predictable processes. The set of admissible $\mathcal{G}$ predictable integrands where the resulting integration is a square integrable semimartingale is:
\[ \Theta(\mathcal{G}) = \{ \xi^{\mathcal{G}} \in L(X, \mathcal{G}) \mid (\xi^{\mathcal{G}} \cdot X)_t \in S^2(\mathbb{P}, \mathcal{G}) \} \]

Note that \( \xi^{V, \mathcal{G}} \in \Theta(\mathcal{G}) \) by definition. The set of signed local martingale measures such that any \( g \in G_T(\Theta(\mathcal{G})) \) is a local martingale (these definitions from Delbaen and Schachermayer [3]) is defined:

\[ D^s(\mathcal{G}) = \{ Z^{\mathcal{G}} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \mid \forall g \in G_T(\Theta(\mathcal{G})), \mathbb{E}[Z^{\mathcal{G}}g] = 0, \mathbb{E}[Z^{\mathcal{G}}] = 1 \} \]

\[ \mathcal{M}^s(\mathcal{G}) = \{ \mathbb{Q}^{\mathcal{G}} \mid d\mathbb{Q}^{\mathcal{G}}/d\mathbb{P} = Z^{\mathcal{G}}, Z^{\mathcal{G}} \in D^s(\mathcal{G}) \} \]

From Lemma 2.1 in Delbaen and Schachermayer [3] the set of signed local martingale measures \( D^s(\mathcal{G}) \neq \emptyset \) is non-empty if the constant 1 is not contained in \( G_T(\Theta(\mathcal{G})) \).

**Assumption 3.1.** \( 1 \notin G_T(\Theta(\mathcal{G})) \) and therefore \( D^s(\mathcal{G}) \neq \emptyset \).

The set of equivalent local martingale measures is the subset of signed local martingale measures that are strictly positive and therefore probability measures.

\[ D^e(\mathcal{G}) = \{ Z^{\mathcal{G}} \in D^s(\mathcal{G}) \mid Z^{\mathcal{G}} > 0 \} \]

\[ \mathcal{M}^e(\mathcal{G}) = \{ \mathbb{Q}^{\mathcal{G}} \mid d\mathbb{Q}^{\mathcal{G}}/d\mathbb{P} = Z^{\mathcal{G}}, Z^{\mathcal{G}} \in D^e(\mathcal{G}) \} \]

The set of signed local martingale measures and equivalent local martingale measures are equal (\( D^e(\mathcal{G}) = D^s(\mathcal{G}), \mathcal{M}^e(\mathcal{G}) = \mathcal{M}^s(\mathcal{G}) \)) if price process \( X \) is \( \mathcal{G} \) continuous (Delbaen and Schachermayer [3]). Therefore by assumption 2.4 \( \mathcal{M}^e(\mathcal{G}) \neq \emptyset \).
3.1.2 The Closed Subspace \( G_T(\Theta(\mathcal{F})) \)

The proper subset (lemma 2.6) \( \Theta(\mathcal{F}) \subset \Theta(\mathcal{G}) \) is defined where all the elements are \( \mathcal{F} \) adapted. Let \( L(X, \mathcal{F}) \) denote the space of all \( X \)-integrable \( \mathcal{F} \) predictable processes:

\[
\Theta(\mathcal{F}) = \{ \xi^\mathcal{F} \in L(X, \mathcal{F}) \mid (\xi^\mathcal{F} \cdot X)_t \in \mathcal{S}^2(\mathbb{P}, \mathcal{F}) \}
\]

Finding the optimal mean square adapted strategy \( \xi^\mathcal{F} \) is a Hilbert space projection from \( L^2(\mathbb{P}) \) onto \( G_T(\Theta(\mathcal{F})) \). Therefore the closure of \( G_T(\Theta(\mathcal{F})) \) in \( L^2(\mathbb{P}) \) is a necessary property. The set of equivalent local martingale measures such that any \( f \in G_T(\Theta(\mathcal{F})) \) is a local martingale is defined:

\[
\mathcal{D}^c(\mathcal{F}) = \{ Z^\mathcal{F} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \mid \forall f \in G_T(\Theta(\mathcal{F})), \mathbb{E}[Z^\mathcal{F} f] = 0, \mathbb{E}[Z^\mathcal{F}] = 1 \}
\]

\[
\mathcal{M}^e(\mathcal{F}) = \{ \tilde{\mathbb{Q}}^\mathcal{F} \mid d\tilde{\mathbb{Q}}^\mathcal{F} / d\mathbb{P} = Z^\mathcal{F}, Z^\mathcal{F} \in \mathcal{D}^c(\mathcal{F}) \}
\]

It is intuitive and true that \( \mathcal{D}^c(\mathcal{F}) \subseteq \mathcal{D}^c(\mathcal{G}) \) and \( \mathcal{M}^e(\mathcal{F}) \subseteq \mathcal{M}^e(\mathcal{G}) \) (Møller [15], Kohlmann, Xiong and Ye [10]) and therefore \( \mathcal{M}^e(\mathcal{F}) \neq \emptyset \).

**Definition** The \( \mathcal{F} \) Variance Optimal Martingale Measure (VOMM) is defined as the equivalent local martingale measure \( \tilde{\mathbb{Q}}^\mathcal{F} \in \mathcal{M}^e(\mathcal{F}) \) such that the associated density \( \tilde{Z}^\mathcal{F} \in \mathcal{D}^c(\mathcal{F}) \) has the minimum \( L^2(\mathbb{P}) \) norm:

\[
\frac{d\tilde{\mathbb{Q}}^\mathcal{F}}{d\mathbb{P}} = \tilde{Z}^\mathcal{F} = \min_{Z^\mathcal{F} \in \mathcal{D}^c(\mathcal{F})} \| Z^\mathcal{F} \|_{L^2(\mathbb{P})} = \min_{Z^\mathcal{F} \in \mathcal{D}^c(\mathcal{F})} \text{Var}[Z^\mathcal{F}]
\]

Similarly, the \( \mathcal{G} \) VOMM \( \tilde{\mathbb{Q}}^\mathcal{G} \) has the minimum \( L^2(\mathbb{P}) \) norm in \( \mathcal{M}^e(\mathcal{G}) \). It is important to note that that, in general, \( \tilde{\mathbb{Q}}^\mathcal{F} \) does not equal \( \tilde{\mathbb{Q}}^\mathcal{G} \) (Møller [15], Kohlmann, Xiong and Ye [10]). In principle, the \( \tilde{\mathbb{Q}}^\mathcal{G} \) may not even be in \( \mathcal{M}^e(\mathcal{G}) \). Therefore the following assumption is required:
Assumption 3.2. The $\mathcal{F}$ Variance Optimal Martingale Measure is also a $\mathcal{G}$ equivalent local martingale measure, $\tilde{\mathbb{Q}}^F \in \mathcal{M}^e(\mathcal{G})$.

Definition A uniformly integrable strictly positive $\mathbb{P}, \mathcal{F}$-martingale $Z > 0$ satisfies the $\mathcal{F}$ reverse Hölder condition, denoted $Z \in \mathcal{R}_p(\mathbb{P}, \mathcal{F})$, if there is a constant $C < \infty$ such that for every stopping time $0 \leq \tau \leq T$ the following relationship exists:

$$
\mathbb{E}\left[ \left( \frac{Z}{Z_\tau} \right)^p \bigg| \mathcal{F}_\tau \right] \leq C \quad p \in (1, \infty)
$$

Assumption 3.3. The VOMM density $\tilde{Z}^F$ satisfies the $\mathcal{F}$ reverse Hölder inequality, $\tilde{Z}^F \in \mathcal{R}_2(\mathbb{P}, \mathcal{F})$.

If the $\mathcal{F}$ VOMM density exists and satisfies the $\mathcal{F}$ reverse Hölder inequality $\tilde{Z}^F \in L^2(\mathbb{P}) \cap \mathcal{M}^e(\mathcal{F}) \cap \mathcal{R}_2(\mathbb{P}, \mathcal{F})$, the closure of $G_T(\Theta(\mathcal{F}))$ in $L^2(\mathbb{P})$ was proved by Delbaen, Monat, Schachermayer, Schweizer and Stricker [2] (theorem 4.1).

3.2 $L^2$ Optimal Strategies

3.2.1 An $L^2$ Optimal Strategy for Martingale X

Let $^P\mathcal{G}$ denote the predictable sigma field of $\mathcal{G}$. The sigma finite Doléans measure $\nu_Q$ and associated product space can be defined using continuous square integrable local martingale $X$ under any equivalent local martingale measure $Q^F \in \mathcal{M}^e(\mathcal{G})$:

$$
(\Omega \otimes [0, T], ^P\mathcal{G} \otimes \mathcal{B} [0, T], \nu_Q); \quad \nu_Q(A) = \int_\Omega \int_0^T 1_A(t, \omega) d\langle X \rangle_t(\omega) dQ^F(\omega).
$$
Lemma 3.4. The explicit conditional expectation of the product space for all \( Q^\theta \in \mathcal{M}^e(\mathcal{G}) \). Let \( A = \mathcal{H} \times B \) where \( \mathcal{H} \subseteq \mathcal{G} \) and \( B \in \mathcal{B}[0,T] \). For any \( \mathcal{G} \) adapted process \( \xi^\theta \in (\Omega \otimes [0,T], \mathcal{G} \otimes \mathcal{B}[0,T], \nu_Q) \) the explicit product space conditional expectation is defined:

\[
\begin{align*}
\xi^{A,Q} &= E^{(X),Q} [\xi^\theta | A] \\
\xi^{A,Q}_t &= 1_B(t) \frac{E^Q [\xi^\theta_t d\langle X \rangle_t | \mathcal{H}_t]}{E^Q [d\langle X \rangle_t | \mathcal{H}_t]} \quad \forall t \in [0,T]
\end{align*}
\]

Proof. Using the definition of conditional expectation, Fubini’s theorem and the definition of the \( \sigma \)-finite Doléans measure:

\[
\begin{align*}
\int_A \xi^\theta d\nu_Q &= \int_{\mathcal{H}} \int_B \xi^\theta_t d\langle X \rangle_t dQ^\theta \\
&= \int_{\mathcal{H}} \int_0^T 1_B(t) E^Q [\xi^\theta_t d\langle X \rangle_t | \mathcal{H}_t] dQ^\theta \\
&= \int_{\mathcal{H}} \int_0^T 1_B(t) \frac{E^Q [\xi^\theta_t d\langle X \rangle_t | \mathcal{H}_t]}{E^Q [d\langle X \rangle_t | \mathcal{H}_t]} E^Q [d\langle X \rangle_t | \mathcal{H}_t] dQ^\theta \\
&= \int_{\mathcal{H}} \int_B 1_B(t) \frac{E^Q [\xi^\theta_t d\langle X \rangle_t | \mathcal{H}_t]}{E^Q [d\langle X \rangle_t | \mathcal{H}_t]} d\langle X \rangle_t dQ^\theta \\
&= \int_A E^{(X),Q} [\xi^\theta | A] d\nu_Q
\end{align*}
\]

For all \( Q^\theta \in \mathcal{M}^e(\mathcal{G}) \) the norm defined on the Hilbert space \( L^2(\Omega \otimes [0,T], \mathcal{G} \otimes \mathcal{B}[0,T], \nu_Q) \) (abbreviated \( L^2(\nu_Q) \)) is isometric (Kunita and Watanabe [12]) to the corresponding norm on \( L^2(\Omega, \mathcal{F}_T, \mathcal{Q}^\theta) \) and the isometry mapping is the Itô integral \( I_X : \xi \mapsto (\xi \cdot X)_T \).
\[ \| \xi^\mathcal{F} \|_{L^2(\nu_0)}^2 = \mathbb{E}^\mathcal{Q}\left[ \int_0^T (\xi^\mathcal{F})^2 d\langle X \rangle \right] = \| (\xi^\mathcal{F} \cdot X)_T \|_{L^2(\mathcal{Q})}^2 = \| I_{X}(\xi^\mathcal{F}) \|_{L^2(\mathcal{Q})}^2 \]

The conditional expectation \( \xi^{\mathcal{F}, \mathcal{Q}} = \mathbb{E}(X)_{\mathcal{Q}}[\xi^\mathcal{F} | \mathcal{F} \times [0, T]] \) is the \( \mathcal{F} \) adapted process that minimizes the norm \( \| \xi^\mathcal{F} - \xi^{\mathcal{F}, \mathcal{Q}} \|_{L^2(\nu_0)} \). Therefore it is immediate from the isometry that \( \xi^{\mathcal{F}, \mathcal{Q}} \) is the \( \mathcal{F} \) adapted integrand that minimizes \( \| (\xi^\mathcal{F} \cdot X)_T - (\xi^{\mathcal{F}, \mathcal{Q}} \cdot X)_T \|_{L^2(\mathcal{Q})} \). This result was obtained in the general multi-variate case by Schweizer [21].

### 3.2.2 An \( L^2 \) Optimal VWAP Strategy for Martingale \( X \)

The explicit product space conditional expectation (eqn 6 in lemma 3.4) can be applied to give the mean square optimal trading strategy for VWAP under any \( \mathcal{Q} \in \mathcal{M}(\mathcal{G}) \) in the associated Hilbert space \( L^2(\Omega, \mathcal{F}_T, \mathcal{Q}) \):

\[ \xi^{Y, \mathcal{F}, \mathcal{Q}}_t = \mathbb{1}_{[0, T]}(t) \frac{\mathbb{E}^\mathcal{Q}[\xi^\mathcal{F}_t d(X)_t | \mathcal{F}_t]}{\mathbb{E}^\mathcal{Q}[d(X)_t | \mathcal{F}_t]} = \mathbb{E}^\mathcal{Q}[\xi^{Y, \mathcal{F}}_t | \mathcal{F}_t] \]  

### 3.2.3 An \( L^2 \) Optimal Strategy for Semimartingale \( X \)

This section is based on the theory of optimal \( L^2 \) hedging for continuous semimartingale processes. The main results of the extensive literature (summarized in Schweizer [23]) are sketched below.

Using the \( \mathcal{F} \) VOMM density \( \tilde{Z}^\mathcal{F} \), the process \( \hat{Z}^\mathcal{F}_t \) can be defined as follows:

\[ \hat{Z}^\mathcal{F}_t = \mathbb{E}^\mathcal{Q}\left[ \frac{d\tilde{Q}^\mathcal{F}}{d\mathbb{P}} | \mathcal{F}_t \right] = \frac{\mathbb{E}[\tilde{Z}^\mathcal{F}^2 | \mathcal{F}_t]}{\mathbb{E}[\tilde{Z}^\mathcal{F} | \mathcal{F}_t]} = \frac{\mathbb{E}[\text{Var}[\tilde{Z}^\mathcal{F}] | \mathcal{F}_t] + 1}{\mathbb{E}[\tilde{Z}^\mathcal{F} | \mathcal{F}_t]} \]

There exists an integrand \( \xi^\mathcal{F} \in \Theta(\mathcal{F}) \) (lemma 2.2 Delbaen and Schachermayer [3]) such that the process \( \hat{Z}^\mathcal{F} \) can be expressed as the integral:
\[ \tilde{Z}_t^\mathcal{F} = \tilde{Z}_0^\mathcal{F} + \int_0^t \xi_t^\mathcal{F} \, dX \]

For any random variable \( H \in L^2(\mathbb{P}) \) the Galtchouk [5], Kunita and Watanabe [12] (GKW) decomposition of \( H \) with respect to martingale \( X \) under VOMM \( \tilde{\mathbb{Q}}^\mathcal{F} \) can be formulated:

\[ H = \mathbb{E}^{\tilde{\mathbb{Q}}}[H] + \int_0^T \xi_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} \, dX + L_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} \]

\( L_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} \) is a \( \tilde{\mathbb{Q}}^\mathcal{F} \) square integrable martingale strongly orthogonal to \( X \) under \( \tilde{\mathbb{Q}}^\mathcal{F} \). The \( \tilde{\mathbb{Q}}^\mathcal{F} \) martingale \( W_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} \) is defined as the conditional expectation of \( H \):

\[ W_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} = \mathbb{E}^{\tilde{\mathbb{Q}}}[H \mid \mathcal{F}_t] = \mathbb{E}^{\tilde{\mathbb{Q}}}[H] + \int_0^t \xi_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} \, dX + L_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} \]

Finally, the mean square optimal integrand \( \xi_t^{H,\mathcal{F}} \) for \( H \) under \( \mathbb{P} \) is given by the following feedback equation:

\[ \xi_t^{H,\mathcal{F}} = \frac{\xi_t^{H,\mathcal{F},\tilde{\mathbb{Q}}}}{Z_t^\mathcal{F}} \left( W_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} - \mathbb{E}^{\tilde{\mathbb{Q}}}[H] - \int_0^t \xi_s^{H,\mathcal{F}} \, dX \right) \]

\[ = \xi_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} - \xi_t^\mathcal{F} \left( \frac{W_0^{H,\mathcal{F},\tilde{\mathbb{Q}}} - \mathbb{E}^{\tilde{\mathbb{Q}}}[H]}{Z_0^\mathcal{F}} - \int_0^t \frac{1}{Z^\mathcal{F}} \, dL_t^{H,\mathcal{F},\tilde{\mathbb{Q}}} \right) \]

### 3.2.4 An \( L^2 \) Optimal VWAP Strategy for Semimartingale \( X \)

The VWAP constant term \( X_0 \) (initial stock price) cannot be modified by the VWAP trader (remark 2.5), therefore it is \( V_T - X_0 \) (the Itô integral term in eqn 1) that is optimized below. The uniqueness of the GKW decomposition and assumption 3.2 (\( \tilde{\mathbb{Q}}^\mathcal{F} \in \mathcal{M}_{c}^e(\mathcal{F}) \)) implies that the mean square optimal
adapted solution for VWAP under $\tilde{Q}^\mathcal{F}$ given by eqn 7 coincides with the GKW decomposition.

$$\mathcal{V}_T - X_0 = \int_0^T \xi_s^{V,\mathcal{F}} dX = \int_0^T \mathbb{E}^{\tilde{Q}}[\xi_s^{V,\mathcal{F}} | \mathcal{F}_s] \, dX_s + L^{V,\mathcal{F},\tilde{Q}}$$

$L^{V,\mathcal{F},\tilde{Q}}$ is the $\mathcal{F}$ measurable difference between traded VWAP and market VWAP $\mathcal{V}_T$ under $\tilde{Q}^\mathcal{F}$:

$$L_t^{V,\mathcal{F},\tilde{Q}} = \mathbb{E}^{\tilde{Q}}\left[ \int_0^t \left( \xi_s^{V,\mathcal{F}} - \mathbb{E}^{\tilde{Q}}[\xi_s^{V,\mathcal{F}} | \mathcal{F}_s] \right) dX_s \bigg| \mathcal{F}_t \right]$$

$$= \int_0^t \left( \mathbb{E}^{\tilde{Q}}[\xi_s^{V,\mathcal{F}} | \mathcal{F}_t] - \mathbb{E}^{\tilde{Q}}[\xi_s^{V,\mathcal{F}} | \mathcal{F}_s] \right) dX_s$$

Using the GKW decomposition above the $\mathcal{F}_t$ conditional expectation $W^{V,\mathcal{F},\tilde{Q}}$ is formulated:

$$W_t^{V,\mathcal{F},\tilde{Q}} = \int_0^t \mathbb{E}^{\tilde{Q}}[\xi_s^{V,\mathcal{F}} | \mathcal{F}_t] \, dX_s$$

Substituting this into equation 8 gives the mean square optimal feedback solution (trading strategy) for VWAP trading with semimartingale $X$:

$$\xi_t^{V,\mathcal{F}} = \mathbb{E}^{\tilde{Q}}[\xi_t^{V,\mathcal{F}} | \mathcal{F}_t] - \frac{\mathcal{F}_t}{Z_t^{\tilde{Q}}} \int_0^t \left( \mathbb{E}^{\tilde{Q}}[\xi_s^{V,\mathcal{F}} | \mathcal{F}_t] - \xi_s^{V,\mathcal{F}} \right) dX_s \quad (9)$$

The mean square optimal VWAP trading strategy equation has an intuitive interpretation. The first term of the equation is the approximation to the $\mathbb{P}$ optimal strategy under the ‘closest’ $\mathcal{F}$ adapted equivalent local martingale measure $\tilde{Q}^\mathcal{F}$. The second term is a feedback error correction term based on updating the accumulated strategy error in the time interval $[0, t)$ using the available information at time $t$. 

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3.3 Mean Variance Optimal Trading

3.3.1 The Minimum Variance VWAP Strategy is the $L^2$ Strategy

The minimum variance solution $\xi^{0,\mathcal{F}}$ is the Markowitz optimal mean-variance utility eqn 3 with a zero risk aversion coefficient, $\lambda = 0$. The optimization of VWAP does not include the optimization of the constant term (remark 2.5). Therefore the optimal mean-square and minimum variance VWAP strategies coincide $\xi^{0,\mathcal{F}} = \xi^{V,\mathcal{F}}$ and eqn 9 above is also the minimum variance solution. Formally, if $\xi^{1,\mathcal{F}} \in \Theta(\mathcal{F})$ is the integrand of the projection of 1 onto $G_T(\Theta(\mathcal{F}))$, then the minimum variance solution is:

$$
\xi^{0,\mathcal{F}} = \xi^{V,\mathcal{F}} + \mathbb{E}^{\hat{Q}} \left[ \int_0^T \xi^{V,\mathcal{F}} dX \right] \xi^{1,\mathcal{F}}
$$

The $\hat{Q}^{\mathcal{F}}$ expectation above is zero since $\hat{Q}^{\mathcal{F}} \in \mathcal{M}(\Theta(\mathcal{F}))$ by assumption 3.2 and therefore $\xi^{0,\mathcal{F}} = \xi^{V,\mathcal{F}}$.

3.3.2 The Expectation of a Minimum Variance VWAP is Zero

The VOMM density $\tilde{Z}^{\mathcal{F}}$ is closely related to the complement of the projection of the constant 1 onto $G_T(\Theta(\mathcal{F}))$. If the operator $\pi$ is defined as the projection operator from $L^2(\mathbb{P})$ onto the orthogonal complement $G_T(\Theta(\mathcal{F}))^\perp$ then:

$$
\tilde{Z}^{\mathcal{F}} = \frac{\pi(1)}{\mathbb{E}[\pi(1)]} = \frac{1}{\mathbb{E}[\pi(1)]} - \int_0^T \frac{\xi^{1,\mathcal{F}}}{\mathbb{E}[\pi(1)]} dX
$$

Hence the expectation of the difference of the market VWAP and the minimum variance VWAP strategy is zero.
0 = \mathbb{E}^Q \left[ \int_0^T (\xi^{\mathcal{V},\mathcal{F}} - \xi^{0,\mathcal{F}}) dX \right] = \mathbb{E} \left[ \tilde{Z}^\mathcal{F} \int_0^T (\xi^{\mathcal{V},\mathcal{F}} - \xi^{\mathcal{V},\mathcal{F}}) dX \right]

= \frac{1}{\mathbb{E}[\pi(1)]} \mathbb{E} \left[ \int_0^T (\xi^{\mathcal{V},\mathcal{F}} - \xi^{\mathcal{V},\mathcal{F}}) dX \right] - \frac{1}{\mathbb{E}[\pi(1)]} \mathbb{E} \left[ \int_0^T \xi^{\mathcal{V},\mathcal{F}} dX \int_0^T (\xi^{\mathcal{V},\mathcal{F}} - \xi^{\mathcal{V},\mathcal{F}}) dX \right]

= \frac{1}{\mathbb{E}[\pi(1)]} \mathbb{E} \left[ \int_0^T (\xi^{\mathcal{V},\mathcal{F}} - \xi^{\mathcal{V},\mathcal{F}}) dX \right]

### 3.3.3 The Variance of a Minimum Variance VWAP

The variance of the minimum variance solution is also the variance of the mean-square solution \(\xi^{\mathcal{V},\mathcal{F}}\) and is derived by Møller [15] (theorem 5.5).

\[
\text{Var} \left[ \int_0^T (\xi^{\mathcal{V},\mathcal{F}} - \xi^{0,\mathcal{F}}) dX \right] = \mathbb{E} \left[ \left( \int_0^T (\xi^{\mathcal{V},\mathcal{F}} - \xi^{\mathcal{V},\mathcal{F}}) dX \right)^2 \right]
\]

\[
= \mathbb{E}^Q \left[ \int_0^T \frac{1}{\tilde{Z}_s} \left( \xi^{\mathcal{V},\mathcal{F}}_s - \xi^{\mathcal{V},\mathcal{F}}_s - \frac{\xi^\mathcal{F}}{\tilde{Z}^\mathcal{F}_s} \int_0^s \left( \xi^{\mathcal{V},\mathcal{F}}_u - \xi^{\mathcal{V},\mathcal{F}}_u \right) dX_u \right)^2 d\langle X \rangle_s \right]
\]

### 3.3.4 Optimal Mean Variance Strategy Applied to VWAP

If the price process is semimartingale under the observed filtration \(\mathcal{F}\), then the trader may wish to exploit expected price movement to ‘beat’ VWAP. A trader can exploit expected price movement for the benefit of his client by adopting a VWAP trading strategy that is riskier than the minimum variance strategy in return for a positive expected return. The optimal mean-variance VWAP trading strategy is derived using the Markowitz quadratic mean-variance utility function (eqn 3) with a risk aversion coefficient \(\lambda > 0\). Noting that the mean-square optimal and minimum variance solutions
coincide $\xi^{0,\mathcal{F}} = \xi^{V,\mathcal{F}}$, the mean-variance optimal strategy can be formulated (Møller [15]) as:

$$\xi^{\lambda,\mathcal{F}} = \xi^{V,\mathcal{F}} - \lambda \left(1 + \text{Var}[\hat{Z}^\mathcal{F}]\right) \xi^{1,\mathcal{F}}$$  \hspace{1cm} (11)

The increases in expected return and variance for a mean variance VWAP trading strategy are proportional to the variance of the density of the Variance Optimal Martingale Measure. There are conceptually two distinct trading strategies, one is the minimum variance VWAP strategy and a ‘price directional’ strategy that uses the properties of the VOMM for additional variance and expected return and is independent of the minimum variance VWAP strategy (McCulloch [14]).

$$\mathbb{E} \left[ \int_0^T (\xi^{V,\mathcal{G}} - \xi^{\lambda,\mathcal{F}}) \, dX \right] = \lambda \text{Var}[\hat{Z}^\mathcal{F}]$$  \hspace{1cm} (12)

$$\text{Var} \left[ \int_0^T (\xi^{V,\mathcal{G}} - \xi^{\lambda,\mathcal{F}}) \, dX \right] = \text{Var} \left[ \int_0^T (\xi^{V,\mathcal{G}} - \xi^{0,\mathcal{F}}) \, dX \right] + \lambda^2 \text{Var}[\hat{Z}^\mathcal{F}]$$  \hspace{1cm} (13)

4 Conclusion

A continuous time model of optimal VWAP trading is considered in this paper. Explicit optimal mean-square ($L^2$) and Markowitz mean-variance VWAP trading strategies are derived where price and volume are semimartingales. These results are novel, they generalize Konishi’s [11] minimum variance solution for a martingale Wiener process in a formal market where trade size is restricted to 1 stock unit.
References


