In the Mean-Variance World there are only Hedge Traders and Price Traders

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Abstract 

The price of a financial claim at a fixed time can represented by a random variable $H$. In an incomplete market $H$ can be approximated by a trading strategy known as minimum variance hedging. Minimum variance hedging can be extended by Markowitz mean-variance optimization where riskier hedging strategies attempt to exploit the expected difference between the hedging strategy and the financial claim. This paper shows that any mean-variance optimal hedging strategy can be decomposed into a linear combination of two conceptually and mathematically different trades. One is the minimum variance hedging strategy and the other is a ‘directional-trade’ that is independent of the hedging strategy and is only dependent on the price of the asset used for hedging. 

Keywords: Optimal Trading Strategies, Minimum Variance Hedging, Price Directional Trading, Variance Optimal Martingale Measure, Markowitz Mean-Variance Optimal Hedging.

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1 Introduction

The value of a financial claim at a fixed time $T$ can represented by a square integrable random variable $H \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ (abbreviated $L^2(\mathbb{P})$). The value of a hedging strategy $\xi$ is an Itô integral $(\xi \cdot X)_T$ with respect to the asset price $X$. If $\Theta$ is the set of feasible hedging strategies, the Markowitz mean variance optimal strategy can be formulated:

$$\xi^\lambda = \max_{\xi \in \Theta} \left[ \mathbb{E} \left[ H - \int_0^T \xi \, dX \right] - \frac{1}{2\lambda} \text{Var} \left[ H - \int_0^T \xi \, dX \right] \right] \quad (1)$$

The minimum variance strategy $\xi^0$ corresponds to the risk aversion coefficient equal to zero, $\lambda = 0$. If a trader wishes to perform a risky hedge with additional return expectation, then $\lambda > 0$. The risky optimal strategy $\xi^\lambda$ can be written as the sum of a minimum variance strategy dependent on $H$ and a ‘directional-trade’ only dependent on the density of the Variance Optimal Martingale Measure (VOMM) (eqn 7). Moreover, the additional expected return (eqn 8) and additional variance (eqn 9) are also only dependent on the VOMM.

Therefore a mean-variance optimal hedging strategy $\xi^\lambda$ can be decomposed into two conceptually and mathematically different trades. One is the minimum variance hedging strategy $\xi^0$ and the other is a ‘directional-trade’ that is only dependent on the VOMM.

2 Mean Variance Optimal Trading Strategies

The stochastic theory of quadratic mean-variance strategies has been studied by Chunli and Karatzas [3], Schweizer [8] [9] [10] and Möller [5] [6] [7] and much of the material in this section has been already been covered in these references, particularly in Möller [5]. However, the section below represents the material specifically with reference to general market trading strategies.

Let $(\Omega, \mathcal{F}_T, \mathcal{F}, \mathbb{P})$ be a stochastic basis with a standard (right continuous and complete) filtration, $\mathcal{F}_T = \{\mathcal{F}_t\}_{0 \leq t \leq T}$. The price process $X$ is a
continuous square integrable semimartingale adapted to $\mathcal{F}$. Let $L(X,\mathcal{F})$ denote the space of all $X$-integrable $\mathcal{F}$ predictable processes. For all integrands $\xi \in L(X,\mathcal{F})$ the integral is notated $G_T(\xi) = (\xi \cdot X)_T$. The set $\Theta$ of admissible $\mathcal{F}$ predictable integrands $\xi$ where the resulting Itô integral is a square integrable semimartingale is defined:

$$\Theta = \{ \xi \in L(X,\mathcal{F}) \mid (\xi \cdot X)_t \in S^2(\mathbb{P},\mathcal{F}) \}$$

**Definition** Let $\mathcal{D}$ denote the set of equivalent local martingale measures of $X$. The Variance Optimal Martingale Measure (VOMM) is defined as the equivalent local martingale measure $\tilde{\mathbb{Q}} \in \mathcal{M}$ such the associated density $\tilde{Z} \in \mathcal{D}$ has the minimum $L^2(\mathbb{P})$ norm:

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = \tilde{Z} = \min_{\tilde{Z} \in \mathcal{D}} \| \tilde{Z} \|_{L^2(\mathbb{P})}$$

In order to project $H \in L^2(\mathbb{P})$ onto $G_T(\Theta)$ the closure of $G_T(\Theta)$ in $L^2(\mathbb{P})$ is a necessary property. This is guaranteed by the assumptions below (see the references for details).

(i) If $1 \notin G_T(\Theta)$ and $X$ is continuous then $\mathcal{D} \neq \emptyset$ (theorem 1.3 and lemma 2.1 in Delbaen and Schachermayer [2]). Therefore the Variance Optimal Martingale Measure (VOMM) exists $\tilde{Z} \in \mathcal{D}$.

(ii) The VOMM satisfies the reverse Hölder inequality $\tilde{Z} \in \mathcal{R}_2(\mathbb{P})$. This assumption ensures the closure of $G_T(\Theta)$ in $L^2(\mathbb{P})$ (theorem 4.1 in Delbaen, Monat, Schachermayer, Schweizer and Stricker [1]).

For any random variable $H \in L^2(\mathbb{P})$ and $\xi \in \Theta$ the mean-variance problem can be solved for a constant $c \in \mathbb{R}$.

$$\xi^c = \min_{\xi \in \Theta} \mathbb{E} \left[ (H - c - G_T(\xi))^2 \right] \quad (2)$$

The integrands of the projections of 1 and $H$ onto $G_T(\Theta)$ are defined:
\[ \xi^1 = \min_{\xi \in \Theta} \mathbb{E} \left[ (1 - G_T(\xi))^2 \right] \]

\[ \xi^H = \min_{\xi \in \Theta} \mathbb{E} \left[ (H - G_T(\xi))^2 \right] \]

Using these definitions, the integrand \( \xi^c \) of the projection of \( H - c \) onto \( G_T(\Theta) \) in eqn 2 can be expressed as:

\[ \xi^c = \xi^H - c \xi^1 \quad (3) \]

The integrand solution \( \xi^c \) was also shown to be mean-variance efficient by Schweizer [8] in the sense that for all \( c \in \mathbb{R} \) then \( \xi^c \in \Theta \) has the property:

\[ \text{Var} \left[ H - G_T(\xi^c) \right] \leq \text{Var} \left[ H - G_T(\vartheta) \right] \]

For all \( \vartheta \in \Theta \) such that:

\[ \mathbb{E} \left[ H - G_T(\xi^c) \right] = \mathbb{E} \left[ H - G_T(\vartheta) \right] \]

The mean-variance efficient property permits an optimal solution for \( c^\lambda \):

\[ c^\lambda = \max_{c \in \mathbb{R}} \left[ \mathbb{E} \left[ H - G_T(\xi^c) \right] - \frac{1}{2\lambda} \text{Var} \left[ H - G_T(\xi^c) \right] \right] \quad (4) \]

The operator \( \pi \) is defined as the projection operator from the Hilbert space \( L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) onto the orthogonal complement \( G_T(\Theta)^\perp \):

\[ \pi(H) = H - G_T(\xi^H) \]

\[ \pi(1) = 1 - G_T(\xi^1) \]

The optimization of \( c^\lambda \) is solved by re-writing the quadratic utility as projections of \( H \) and \( 1 \):
\[ E[H - G_T(\xi^c)] = E[\pi(H)] + c E[1 - \pi(1)] \]

\[ E\left[(H - G_T(\xi^c))^2\right] = E[\pi(H)^2] + 2c E[\pi(H) - \pi(1)] + c^2 E[(1 - \pi(1))^2] \]

This can be simplified by noting that \((1 - \pi(1)) \in G_T(\Theta)\) and \(\pi(H) \in G_T(\Theta)\) and therefore \(E[(1 - \pi(1)) \pi(H)] = 0\). Also \(E[(1 - \pi(1))] = 0\) and therefore \(E[\pi(1)] = E[\pi(1)^2]\) and similarly \(E[1 - \pi(1)] = E[(1 - \pi(1))^2]\):

\[ E\left[(H - G_T(\xi^c))^2\right] = E[\pi(H)^2] + c^2 E[1 - \pi(1)] \]

\[ E[H - G_T(\xi^c)]^2 = E[\pi(H)]^2 + 2c E[\pi(H)] E[1 - \pi(1)] + c^2 (E[1 - \pi(1)])^2 \]

Thus the quadratic utility function can be specified as a function of orthogonal projections and the constant \(c\):

\[ E[H - G_T(\xi^c)] - \frac{1}{2\lambda} \text{Var}[H - G_T(\xi^c)] \]

\[ = E[\pi(H)] + c E[1 - \pi(1)] - \frac{1}{2\lambda} \left( E[\pi(H)^2] + c^2 E[1 - \pi(1)] \right. \]

\[ \left. - (E[\pi(H)])^2 - 2c E[\pi(H)] E[1 - \pi(1)] - c^2 (E[1 - \pi(1)])^2 \right) \]

The expression is quadratic in \(c\) and dropping all terms without \(c\) gives:

\[ c E[1 - \pi(1)] - \frac{1}{2\lambda} \left( c^2 E[1 - \pi(1)] - 2c E[\pi(H)] E[1 - \pi(1)] - c^2 (E[1 - \pi(1)])^2 \right) \]

(5)
Note that the second derivative of the above with respect to \( c \) is non-positive if \( 0 \leq \mathbb{E}[\pi(1)] \leq 1 \):

\[
\left( \frac{\mathbb{E}[\pi(1) - 1]}{\lambda} \right) \mathbb{E}[\pi(1)] \leq 0 \implies 0 \leq \mathbb{E}[\pi(1)] \leq 1
\]

This condition is true since \( \mathbb{E}[\pi(1)] = \mathbb{E}[\pi(1)^2] \geq 0 \) and \( \mathbb{E}[1 - \pi(1)] = \mathbb{E}[(1 - \pi(1))^2] \geq 0 \). Therefore the equation above is maximized if the first derivative of eqn 5 with respect to \( c \) is zero:

\[
\mathbb{E}[1 - \pi(1)] - \frac{1}{2\lambda} \left( 2c \mathbb{E}[1 - \pi(1)] - 2\mathbb{E}[\pi(H)] \mathbb{E}[1 - \pi(1)] - 2c (\mathbb{E}[1 - \pi(1)])^2 \right) = 0
\]

Solving for \( c^\lambda \) gives:

\[
c^\lambda = \frac{\mathbb{E}[\pi(H)]}{\mathbb{E}[\pi(1)]} + \lambda
\]

### 2.1 The Minimum Variance Solution

From the utility function eqn 4 and eqn 6 for \( c^\lambda \) it is clear that the constant \( c^0 \) of the minimum variance solution corresponds to \( \lambda = 0 \) in eqn 1.

\[
c^0 = \frac{\mathbb{E}[\pi(H)]}{\mathbb{E}[\pi(1)]}
\]

Define closed subspace \( K = \text{span}\{1, G_T(\Theta)\} \subset L^2(\mathbb{P}) \). Then the VOMM density \( \check{Z} \in K \) and is closely linked with the projection of the constant 1 onto \( G_T(\Theta) \):

\[
\check{Z} = \frac{\pi(1)}{\mathbb{E}[\pi(1)]} = \frac{1}{\mathbb{E}[\pi(1)]} - \int_0^T \frac{\xi^1}{\mathbb{E}[\pi(1)]} dX
\]

\[
\text{Var}[\check{Z}] = \frac{1}{\mathbb{E}[\pi(1)]} - 1
\]
Hence the minimum variance constant $c^0$ can be expressed as the expectation of $H$ with respect to the VOMM:

$$
c^0 = \frac{\mathbb{E} \left[ \pi(H) \right]}{\mathbb{E} \left[ \pi(1) \right]} = \frac{\mathbb{E} \left[ \pi(1)H \right]}{\mathbb{E} \left[ \pi(1) \right]} = \mathbb{E}_\mathbb{Q} [H]
$$

The minimum variance integrand $\xi^0 \in \Theta$ can be expressed (corollary 16, Schweizer [9]):

$$
\xi^0 = \xi^H - \mathbb{E}_\mathbb{Q} [H] \xi^1
$$

The variance of the minimum variance solution is:

$$
\text{Var} \left[ H - G_T(\xi^0) \right] = \mathbb{E} \left[ \pi(H)^2 \right] - \frac{\mathbb{E} \left[ \pi(H) \right]^2}{\mathbb{E} \left[ \pi(1) \right]} \tag{8}
$$

The expected return of the minimum variance solution is:

$$
\mathbb{E} \left[ H - G_T(\xi^0) \right] = \frac{\mathbb{E} \left[ \pi(H) \right]}{\mathbb{E} \left[ \pi(1) \right]} = \mathbb{E}_\mathbb{Q} [H] \tag{9}
$$

### 2.2 The Projection of $H$ onto span{1, $G_T(\Theta)$}

The constant $c^K$ of the projection of any $H \in L^2(\mathbb{P})$ onto $K = \text{span}\{1, G_T(\Theta)\}$ is readily calculated by modifying eqn 2 and applying eqn 3:

$$
c^K = \min_{c \in \mathbb{R}} \mathbb{E} \left[ \left( H - (G_T(\xi^c) - c) \right)^2 \right] = \min_{c \in \mathbb{R}} \mathbb{E} \left[ \left( \pi(H) - c \pi(1) \right)^2 \right]
$$

As above, by dropping the terms without $c$ and taking the derivative with respect to $c$, the minimum $c^K$ is readily found (corollary 16, Schweizer [9]):

$$
c^K = \frac{\mathbb{E} \left[ \pi(H) \right]}{\mathbb{E} \left[ \pi(1) \right]} = \mathbb{E}_\mathbb{Q} [H] \tag{10}$$
Theorem 2.1. Any $H \in L^2(\mathbb{P})$ has the following unique decomposition:

$$H = \mathbb{E}_\tilde{Q}[H] + G_T(\xi^K) + L^H$$

Where $\mathbb{E}[L^H] = 0$, $\mathbb{E}[G_T(\xi^K)L^H] = 0$ and $\xi^K = \xi^H - \mathbb{E}_\tilde{Q}[H]\xi^1$.

Proof. The Hilbert space projection theorem and eqn 10. $\square$

2.3 The Mean Variance Utility Solution

A non-zero risk aversion coefficient $\lambda > 0$ in eqn 1 produces a solution that trades off increased variance for increased expected return. The associated mean-variance optimal integrand $\xi^\lambda \in \Theta$ can formulated (Møller [5] [6]):

$$\xi^\lambda = \xi^H - \frac{\mathbb{E}[\pi(H)]}{\mathbb{E}[\pi(1)]} \xi^1 = \xi^0 - \lambda \left(1 + \text{Var}[\tilde{Z}]\right) \xi^1 \quad (11)$$

The expected return and variance terms of a non-zero risk aversion coefficient $\lambda > 0$ are independent of the minimum variance approximation of $H$ and depend only on the VOMM:

$$\mathbb{E} \left[ H - G_T(\xi^\lambda) \right] = \mathbb{E} \left[ H - G_T(\xi^0) \right] + \lambda \text{Var}[\tilde{Z}] \quad (12)$$

$$\text{Var} \left[ H - G_T(\xi^\lambda) \right] = \text{Var} \left[ H - G_T(\xi^0) \right] + \lambda^2 \text{Var}[\tilde{Z}] \quad (13)$$

Remark 2.2. The risk aversion coefficient ($\lambda \geq 0$) terms for increased variance and expected return are independent of the random variable $H$ and only depend on the Variance Optimal Martingale Measure (VOMM). This general result shows that there are conceptually two distinct trading strategies. One minimizes the variance between $H$ and $G_T(\Theta)$ and one uses the expected mean-variance properties of the VOMM for additional variance and expected return.
References


