A Regression Model of True Spreads in Limit Order Markets *

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Abstract

True spreads are not directly observable and represent the continuous demand and supply schedule for stock liquidity by heterogeneously informed market participants. Observed spreads are true spreads quantized by minimum market tick size. A regression model of true spreads is developed using spread data from a pure limit order electronic exchange. True spreads are modelled as a continuous positive distribution parameterized by stock turnover and volatility. Nominal stock price is not a significant explanatory variable in modelling true spreads. Nominal stock price is shown to be a significant explanatory variable of observed spreads only as an artifact of minimum tick size.

Keywords: True Spread, Censored Spread, Observed Spread, Tick Size, Exchange Policy

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1 Introduction

Stock exchange trading rules specify a minimum price variation known as tick-size. The exchange tick size therefore implies a minimum bid-ask spread. Ball and Chordia [3] examine spreads on the New York Stock Exchange (NYSE) and find that, for high turnover stocks, minimum price variation is a substantial component of observed spread. They call the component of spread not due to minimum price variation the ‘true’ spread. Harris [9] also examines spreads on the NYSE and finds that minimum tick size is a component of observed spreads. This research answers an important question raised by Harris [9] (page 176) in the conclusion of his paper:

Is stock price only a determinant of observed spreads through the minimum price variation effect on relative spreads?

This research answers this question for spreads on a pure limit order market, the Australian Stock exchange (ASX). Stock price is only a determinant of observed spreads through the minimum price variation effect on relative spreads.

This is a powerful and intuitive result. Turnover is a determinant of spreads because it is a proxy for the scarcity of stock liquidity (Demsetz [6]). Volatility is a determinant of spread because of the extra risk taken by a limit order trader when providing liquidity for a volatile stock (Copeland and Galai [5]). But it is intuitive and reasonable that price is not a determinant of true spread because liquidity providers and demanders are neutral about the nominal stock price at which liquidity is supplied. For example, in negotiating the cost of supplying $1m of stock, liquidity providers and demanders are neutral as whether that stock liquidity is supplied as $1 \times 1,000,000$ stock units or $100 \times 10,000$ stock units.

Cross sectional OLS regression models of relative spreads using observed spreads have strong explanatory power (Harris [9], Stoll [12]). However these models have been shown to be mis-specified by Ball and Chordia [3] because they do not explicit account for minimum tick size rounding. Harris [8] [9] suggested a likelihood regression for finding a parametric distribution that matches the continuous true spread distribution and this technique is further developed in this paper. The unobserved continuous true spread
distribution is assumed to have a constant shape for all stocks and the the scale of the distribution (the distribution mean) is regressed as a function of stock Turnover, Volatility and Price. Price is found to be not be a significant in determining the scale (mean) of the unobserved continuous true spread. The ‘censored’ spread is obtained as a definite integral of the underlying continuous true spreads as partitioned by tick size. Censored spread is a predictive and unbiased estimate (section 3.6) of observed spread obtained by explicitly modelling the effect of minimum tick size on true spreads.

Figure 1: Observed spreads (blue triangles) for stock NAB (National Australia Bank, July 2001 VWAP $32.63, average daily turnover $78.9 million, annualized standard deviation 30.5% ) can only be observed on half-tick (0.5 cent) intervals; at 0.5 cents, 1.0 cent, 1.5 cents etc. The true spread is obscured by minimum tick size and is hypothesized to be a continuous distribution (log normal) on the positive half-line. The mean (scale) of the distribution is modelled as a likelihood regression of stock turnover and volatility (standard deviation). Price is not a significant as a regressor of the unobserved continuous spread. The censored spread (the magenta circles) is the continuous spread distribution partitioned by minimum tick size and is a predictive and unbiased estimate of observed spreads.
2 Spreads and Minimum Tick Size

2.1 The Spreads Data

The Australian Securities Exchange (ASX) market data used in this paper is very rich and allows the limit order book evolution and generated trades to be replayed with complete accuracy. In particular, the complete depth of the limit order queues are known before each trade and traded spread calculations are unambiguous and accurate. For all stocks (non ordinary stock securities such as warrants, hybrid debt-equity, etc. were excluded) trading on the ASX in July 2001 with a stock price equal to or greater than $0.50 (all stocks with a minimum 1 cent tick size for the data period) the actual spread relative to the bid-ask mid-point price paid by pure market orders (buyers and sellers) was recorded in $50 \times 0.5$ cent spread bins. Each spread bin corresponding to a particular spread size recorded the total volume of the market order trades at that spread size. The first bin corresponded to the minimum observable tick size of 0.5 cents, the next bin 1.0 cent and so on. The final bin, $50 \times 0.5$ cents = 25 cents, contained the sum of the volume of all spreads $\geq 25$ cents. Only stocks that had at least 1 spread observation per day (1 pure market trade per day) for the 20 business days of market data during July 2001 were analyzed, 249 stocks had spread data for the 20 business days. For each of the 249 stocks, Turnover, Price and Volatility were calculated using the following definitions:

(i) \textit{Price} is the VWAP (Volume Weighted Average Price) price of the stock for all on-market trades in July 2001.

(ii) \textit{Turnover} is the turnover of all on-market trades in July 2001 divided by the 20 business days to give an average daily turnover.

(iii) \textit{Volatility} is calculated as a weekly standard deviation from the previous 52 weekly prices. Weekly prices were used rather than daily prices to minimize serial price correlations in low turnover stocks which would (downward) bias the volatility estimates.
2.2 Measuring Spreads

When measuring spreads, the spread cost paid by market order traders to initiate a trade and execute against limit order traders is of primary interest. It is intuitive to measure the spread as the absolute difference between the transaction price and the actual stock price.

\[
\text{spread} = |\text{ActualPrice} - \text{TransactionPrice}|
\]

The actual stock price \( P_s \) cannot, in general, be observed but lies somewhere between the current quoted bid and ask price. If it is assumed that actual stock prices are uniformly distributed or symmetrically distributed about the mid-point of the bid and ask price, then the expectation of the actual price will be the mid-point of the bid and ask price. Thus a practical spread cost measurement is the absolute difference between the current mid-point of the bid and ask price and the transaction price. Effective spread \( S_e \) is defined as the difference between the trade price \( P_t \) and the mid-point quoted price \( \frac{\text{bid} + \text{ask}}{2} \). Effective spread has been chosen as the spread measure in investigations into market spread on the Toronto stock exchange by Bacidore [2], the NASDAQ exchange by Barclay et al [4] and Ball and Chordia [3], Harris [9] on the NYSE.

\[
S_e = |P_s - P_t| \approx \left| \frac{\text{ask} + \text{bid}}{2} - P_t \right| \quad (1)
\]

Relative effective spread \( S_r \) is effective spread normalized by dividing by stock price \( P_s \) and, following market convention, is scaled by 10,000 so that the relative spread is expressed in ‘basis points’ (1/100ths of a percent).

\[
S_r = 10000 \frac{S_e}{P_s} \approx 10000 \left| 1 - \frac{2P_t}{\text{ask} + \text{bid}} \right| \quad (2)
\]
2.3 Minimum Relative Spread Cost

For all stocks in this study, the minimum tick size is 1 cent and therefore the minimum spread is $0.005. The minimum relative spread cost in basis points, $S_{\text{min}}$, is given by the following relationship to stock price, $P_s$ in dollars.

$$ S_{\text{min}} = \frac{50}{P_s}; \quad \log S_{\text{min}} = \log 50 - \log P_s $$

(3)

Figure 2 shows the minimum cost of crossing the spread and executing a market order rapidly increases for lower priced stocks. The cost of executing a market order must be on or above the minimum relative cost line in figure 2. The price of a stock determines the minimum relative cost of executing a market order.

![Figure 2: Minimum Relative Spread Cost as a Function of Price](image-url)
2.4 OLS Regressions are Misspecified

Both Harris [9] and Stoll [12] develop cross sectional OLS regression models where relative spreads are regressed against price, market activity (trades per day), turnover, market capitalization and volatility. These OLS regression models have strong explanatory power when describing spreads on the NYSE and NASDAQ stock markets with Stoll stating (Stoll [12], page 1481) ‘few empirical relations in finance are this strong’. However these models have been shown to be mis-specified by Ball and Chordia [3] because they do not explicitly account for minimum tick size rounding.

Figure 3: Daily observed relative spreads against stock price for top decile (highest) turnover stocks. Observed spreads are censored against the minimum spread line (black line) for lower priced stocks.
2.5 A Predictive but Misspecified OLS Regression

A modified parsimonious version of the regression equation specified by Stoll [12] was estimated. The regression estimates the logarithm (all logs in this paper are base 10) of observed relative spread \( S_r \) using the logarithm of turnover \( T_s \), Price \( P_s \) and Volatility \( \sigma_s \) as explanatory variables. Additional explanatory variables such as Trade Count and Market Capitalization did not add significant information to the regression. This regression equation was also found to be predictive by Frino and Aitken [1] on ASX spread data under an earlier 1996 minimum tick-size schedule (the minimum tick-size schedule on the ASX has progressively decreased over time).

The regression was tested for heteroscedasticity (‘Hetero-X’, White [13]) and heteroscedasticity was confirmed (\( F(9, 235) = 32.55 \)). Heteroscedastic consistent t-statistics for the explanatory variables were generated using the Jack-Knife estimator (‘t-JHCSE’, White and MacKinnon [10]) and all explanatory variables are significant at very high confidence levels (>99.99%). Partial \( R^2 \) statistics (Nachtsheim et al [11]) for the explanatory variables are also calculated.

\[
\log S_r = A_0 + A_1 \log T_s + A_2 \log P_s + A_3 \log \sigma_s + \varepsilon_s
\]  \hspace{1cm} (4)

<table>
<thead>
<tr>
<th>Intercept</th>
<th>( \log ) Turnover</th>
<th>( \log ) Price</th>
<th>( \log ) Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficients</td>
<td>3.499</td>
<td>-0.268</td>
<td>-0.354</td>
</tr>
<tr>
<td>t-JHCSE</td>
<td>23.663</td>
<td>-9.878</td>
<td>-7.160</td>
</tr>
<tr>
<td>Partial ( R^2 )</td>
<td>0.666</td>
<td>0.547</td>
<td>0.179</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.911</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hetero-X ( F(9, 235) = 32.55 ) (( p = 0.000 ))</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The ASX Parsimonious Relative Spread Regression Model (n=249)
3 A Model of True Spreads

The previous section showed that OLS models of spreads, although highly predictive, were an incorrect description of spread behaviour because of data censoring. In this section the spread model first proposed by Harris [8] [9] is generalized and developed as a model of spreads on the ASX. This model assumes that relative spreads form a continuous distribution and that the observed discrete half-tick size relative spreads are a result of partitioning this continuous distribution into discrete half-tick size increments. Harris used maximum log-likelihood techniques to estimate the underlying parametric relative spread distribution and this technique is further developed by extending the size of the discrete half-tick size distribution observed to 50 observations and examining different candidate parametric relative spread distributions (Harris only examined the Gamma distribution).

3.1 Parametric Continuous True Spread Distributions

Assume a cumulative probability for relative true spread, $\Phi(r; \Lambda)$, where $\Lambda$ is a vector of distributional parameters (possibly scalar). The actual effective spreads paid by market order traders for stock $S$ are observed and summed on a $K \times 0.5$ cent grid (in this paper, $K = 50$). The $K$ spread step probabilities can be written recursively in terms of the cumulative probability distribution of the relative true spread.

$$
Pr(n = 1) = \Phi(\pi(1, P_s); \Lambda) \\
Pr(n = 2) = \Phi(\pi(2, P_s); \Lambda) - Pr(n = 1) \\
\vdots \\
Pr(n = K - 1) = \Phi(\pi(n - 1, P_s); \Lambda) - Pr(n = K - 2) \\
Pr(n \geq K) = 1 - Pr(n = K - 1)
$$

The function $\pi(n, P_s)$ is a mapping function from the continuous (but unobserved) spread distribution to the observed discrete half-tick sized spreads. The mapping function does this by mapping the observed $n$th 0.5 cent spread step onto a continuous spread that represents the mid-point of $n$ spread steps.
and $n+1$ spread steps. An unobserved continuous spread, $r$, will be observed as an actual spread of $n \times 0.5$ cents if the unobserved continuous spread is in the following interval.

$$\pi(n-1, P_s) \leq r < \pi(n, P_s)$$

The arithmetic mid-point between two observed spreads is used to determine the boundary for a continuous unobserved spread to move from an observed spread step size of $n \times 0.5$ cent steps to $(n + 1) \times 0.5$ cent steps. Therefore $\pi(n)$ is defined as follows (in basis points):

$$\pi(n, P_s) = \frac{50(n + 0.5)}{P_s}, \quad \pi(0, P_s) = 0 \quad (6)$$

### 3.1.1 The Loglikelihood Function

A multinomial likelihood function (constant omitted) can be generated for $M$ stocks with $K$ actual spread step observations (each observation is $m_{j,k}$) for each stock.

$$L(\Lambda) = \prod_{j=1}^{M} \prod_{k=1}^{K} \left[ Pr(n = k) \right]^{m_{j,k}} \quad (7)$$

The consequent log-likelihood function is defined as:

$$\ell(\Lambda) = \ln L(\Lambda) = \sum_{j=1}^{M} \sum_{k=1}^{K} m_{j,k} \ln Pr(n = k) \quad (8)$$

If $f_j(k)$ is introduced as the observed relative frequency of actual spread step $k$ for the $j$th stock and substituting equations 5 into the log-likelihood equation 8.
\[
M_j = \sum_{k=1}^{K} m_{j,k} \quad f_j(k) = \frac{m_{j,k}}{M_j}
\]

\[
\ell(\Lambda) = \sum_{j=1}^{M} M_j \left[ f_j(1) \ln[\Phi(\pi(1, P_j); \Lambda)] + f_j(2) \ln[\Phi(\pi(2, P_j); \Lambda) - \Phi(\pi(1, P_j); \Lambda)] + \cdots + f_j(K - 1) \ln[\Phi(\pi(K - 1, P_j); \Lambda) - \Phi(\pi(K - 2, P_j); \Lambda)] + f_j(K) \ln[1 - \Phi(\pi(K - 1, P_j); \Lambda)] \right]
\]

In order to solve for the relative spread distribution, a distributional class, \(\Phi(r; \Lambda)\), needs to be selected with a mean of \(S_r\) for different values of the shape parameter vector \(\Lambda\) and supported on the non-negative numbers, \(r \in [0, \infty)\). Then \(\Lambda\) can be solved by maximizing the log-likelihood equation 9.

\[
\Lambda^0 = \arg \max_{\Lambda} [\ell(\Lambda)]
\]

### 3.2 Parameter Estimation of Continuous True Spread Distributions

Harris [8] [9] uses the Gamma distribution to model the spread distribution by solving equations 9 and 10. The Gamma distribution can be parameterized by shape \(\lambda\) and scale \(\beta\) parameters:

\[
\Phi(r; \lambda, \beta) = \frac{1}{\beta^\lambda \Gamma(\lambda)} \int_0^r x^{\lambda - 1} e^{-x/\beta} \, dx \quad r \in [0, \infty)
\]
\[ E[\Phi(r; \lambda, \beta)] = \lambda \beta \]

If the true spread, \( S^\text{true}_r \) is defined as the mean of the continuous spread distribution.

\[ S^\text{true}_r = E[\Phi(r; \lambda, \beta)] = \lambda \beta \]

The scale parameter \( \beta \) can be modified to be a function of the shape parameter \( \lambda \) and the mean uncensored spread \( S^\text{true}_r \). With this specification, the shape parameter \( \lambda \) can be constant and the scale parameter \( \beta \) varied so that the distributional mean is always \( S^\text{true}_r \).

\[ \beta = \frac{S^\text{true}_r}{\lambda} \]

The true spread \( S^\text{true}_r \) can be modelled explicitly as a function of the Turnover, Price and Volatility of a stock. The same functional form as the parsimonious regression (eqn 4) is chosen.

\[ \log S^\text{true}_r = B_0 + B_1 \log T_s + B_2 \log P_s + B_3 \log \sigma_s \quad (11) \]

Therefore the scale parameter of the Gamma distribution is written as a function of the shape parameter and the parameters of the regression equation (eqn 11).

\[ \beta = 10^{-\left(B_0 + B_1 \log T_s + B_2 \log P_s + B_3 \log \sigma_s\right)} \frac{1}{\lambda} \]

Substituting the re-parameterized Gamma distribution into the log-likelihood equation 9 produces a model with 5 parameters to solve, the shape of the continuous distribution \( \lambda \) and the scale of the continuous distribution expressed as a regression on the stock Turnover \( T_s \), Price \( P_s \) and Volatility \( \sigma_s \).
$$\Lambda^0 = \{\lambda^0, B^0_0, B^0_1, B^0_2, B^0_3\} = \text{arg min}\ [ -\ell(\lambda, B_0, B_1, B_2, B_3) ]$$

The Gamma distribution suggested by Harris [8] [9] has a rich range of distributional shapes but there are other shape and scale distributions defined on the positive half-line and suggested by inspection of the binned spread data that can be used in the general log-likelihood model developed in equation 9. Alternatives are the Weibull distribution and the Log-Normal distribution. The Exponential distribution is also modelled because the results of modelling the Gamma and Weibull distributions (table 2) produced shape parameters ($\lambda = 1$) close to this distribution.

The Exponential distribution is dependent on a single scale parameter $\beta$.

$$\Phi(r; \beta) = 1 - \exp[-\beta r]$$

$$E[\Phi(r; \beta)] = \frac{1}{\beta} = S_{r}^{\text{true}}$$

Therefore the scale parameter of the exponential distribution is written as a function of the parameters of the regression equation only.

$$\beta = 10^{-(B_0 + B_1 \log T_s + B_2 \log P_s + B_3 \log \sigma_s)}$$

The Weibull distribution is parameterized by a shape $\lambda$ and scale $\beta$ parameter.

$$\Phi(r; \lambda, \beta) = 1 - \exp[-(\frac{r}{\beta})^\lambda]$$

$$E[\Phi(r; \lambda, \beta)] = \beta \Gamma\left(\frac{\lambda + 1}{\lambda}\right) = S_{r}^{\text{true}}$$
$$\beta = 10^{(B_0 + B_1 \log T_s + B_2 \log P_s + B_3 \log \sigma_s)} \left[ \frac{\Gamma(\frac{\lambda + 1}{\lambda})}{\lambda} \right]^{-1}$$

The Log-Normal distribution is parameterized by a shape $\lambda$ and scale $\beta$ parameter as follows:

$$\Phi(r; \lambda, \beta) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\ln r - \beta}{\sqrt{2} \lambda} \right) \right], \quad \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$$E[\Phi(r; \lambda, \beta)] = \exp \left[ \beta + \frac{\lambda^2}{2} \right] = S_r^{\text{true}}$$

$$\beta = \ln(10)(B_0 + B_1 \log T_s + B_2 \log P_s + B_3 \log \sigma_s) - \frac{\lambda^2}{2}$$

The minimum number of spread observations for all 249 stocks was 70 (median spread observations 731, max spread observations 39,831) and this was used as the $M_j$ term in the log-likelihood function (eqn 9) for all stocks ($M_j = 70, \forall j$). This results in conservative parameter interval estimates and avoids high turnover stocks being over-influential in the ML regression. The proportion of total volume for each spread bin, $f_j(k)$, was calculated by dividing the volume summed in the $k$th spread bin by the total volume executed as pure market orders for the stock. Equation 10 was solved numerically and convergence to optimal values was prompt and consistent with no evidence of local minima.

### 3.3 Log-likelihood Regression Results

The results of fitting the parametric distributions to the binned spreads data are tabulated in table 2. The true relative spread regression model is robust and insensitive to the particular distribution used. The Weibull and

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1 The optimization software was written in the specialist econometrics programming language OX developed by Jurgen A. Doornik [7]. For purely practical reasons associated with the optimization software, the optimization is actually performed by minimizing the negative value of $\ell(\Lambda)$. 

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Gamma distributions both have estimated optimal shape parameters close to 1. A shape parameter of 1 defines the exponential distribution for both these distributions. The Log-Normal distribution had the highest maximum likelihood value, \( \ell(\Lambda^0) = -12268 \), and was the best regression fit.

The estimated true relative spread regressions have larger coefficients for log Turnover and log Volatility than the misspecified OLS spread regression (eqn 4, table 1), but the main difference is the insignificant coefficient for log Price, this coefficient is not significantly different from zero for any of the log-likelihood regressions. There is no significant relationship between stock price and true spreads.
<table>
<thead>
<tr>
<th>Distribution</th>
<th>$\lambda$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LogNormal</td>
<td>0.834</td>
<td>-0.420</td>
<td>-0.010</td>
<td>0.493</td>
<td>4.443</td>
</tr>
<tr>
<td>Std Error</td>
<td>0.0106</td>
<td>0.0081</td>
<td>0.0146</td>
<td>0.0225</td>
<td>0.0487</td>
</tr>
<tr>
<td>$\ell(\Lambda^0)$</td>
<td>-12268</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gamma</td>
<td>1.127</td>
<td>-0.415</td>
<td>-0.014</td>
<td>0.469</td>
<td>4.368</td>
</tr>
<tr>
<td>Std Error</td>
<td>0.0286</td>
<td>0.0076</td>
<td>0.0134</td>
<td>0.0209</td>
<td>0.0457</td>
</tr>
<tr>
<td>$\ell(\Lambda^0)$</td>
<td>-12289</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Weibull</td>
<td>1.031</td>
<td>-0.418</td>
<td>-0.0067</td>
<td>0.473</td>
<td>4.382</td>
</tr>
<tr>
<td>Std Error</td>
<td>0.0126</td>
<td>0.0079</td>
<td>0.0139</td>
<td>0.0215</td>
<td>0.0472</td>
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<tr>
<td>$\ell(\Lambda^0)$</td>
<td>-12296</td>
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<td></td>
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<tr>
<td>Exponential</td>
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<td>0.004</td>
<td>0.480</td>
<td>4.407</td>
<td></td>
</tr>
<tr>
<td>Std Error</td>
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<td>0.0136</td>
<td>0.0219</td>
<td>0.0475</td>
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</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OLS</td>
<td>-0.268</td>
<td>-0.354</td>
<td>0.233</td>
<td>3.499</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: True relative spread distribution parameters (50 x 0.5 cent bins; n=249 Stocks) for different parametric distributions. Also, for comparison, the coefficients of the OLS regression (equation 4, table 1).
3.4 The Tick Size Censoring Function

Observed spreads can be modelled as a half a tick size partitioning of the underlying continuously distributed true spread. This estimate of observed spreads is termed censored spreads. The censoring function is simply the probability weighted sum of the underlying continuously distributed true spread partitioned into discrete observed spreads, conceptually:

\[ S_{r_{\text{observed}}} \approx S_{r_{\text{censored}}} = \text{Censor}(S_{r_{\text{true}}}, \text{TickSize}, \text{Price}) \]  \hspace{1cm} (12)

Note that only the censoring function is affected when tick size is varied through stock exchange policy change. A tick size variable, \( \rho \), is introduced when defining the censoring function this enables the censoring function to be parameterized for different tick sizes. Thus the effect of varying exchange policy on tick size on observed spreads can be investigated.

The censored spread over the observed discrete half tick size distribution is calculated by summing over the probability that the continuous true spread will be in the \( k \)th half spread bin, \( Pr(k, \rho, P_s) \), by the relative spread size of the \( k \)th bin, \( \mu(k, \rho, P_s) \).

\[ S_{r_{\text{censored}}} = \sum_{k=1}^{\infty} Pr(k, \rho, P_s) \mu(k, \rho, P_s) \]  \hspace{1cm} (13)

Formally, the sum in equation 13 is over an infinite number of terms because the uncensored continuous distribution of relative spreads, \( \Phi(r; \Lambda) \), is defined over the whole non-negative real interval \([0, \infty)\). In practise, an arbitrarily accurate approximation can be achieved by truncating the summation when the remaining terms of the infinite sum become small.

The half tick size mid-point function (eqn 6) is augmented by an explicit tick size argument, \( \rho \) - tick size in dollars. (the mid-point is in basis points).

\[ \pi(n, \rho, P_s) = \frac{5000 \rho (n + 0.5)}{P_s}, \quad \pi(0, \rho, P_s) = 0 \]
Let $\Phi(r; \Lambda)$ be the true spread cumulative distribution function for a vector of known spread distribution parameters $\Lambda$. The probability of a true relative spread being observed at the $k$th half-tick size bin, $Pr(k, \rho, Ps)$, is simply the cumulative spread distribution at the $k$th spread bin mid-point less the cumulative spread distribution at the $(k - 1)$th mid-point.

$$Pr(k, \rho, Ps) = \Phi(\pi(k, \rho, Ps); \Lambda) - \Phi(\pi(k - 1, \rho, Ps); \Lambda)$$  \hfill (14)

The relative spread (in basis points) of the $k$th half tick size, parameterized for market tick size $\rho$ (in dollars) is:

$$\mu(k, \rho, Ps) = \frac{5000 k \rho}{Ps}$$  \hfill (15)

### 3.4.1 The Censoring Function

Using the functions defined above the complete censoring function is defined:

$$S_r^{\text{censored}} = \sum_{k=1}^{\infty} Pr(k, \rho, Ps) \mu(k, \rho, Ps)$$

$$= \sum_{k=1}^{\infty} \left[ \Phi(\pi(k, \rho, Ps); \Lambda) - \Phi(\pi(k - 1, \rho, Ps); \Lambda) \right] \frac{5000 k \rho}{Ps}$$  \hfill (16)
3.5 True Spreads Compared to Censored Spreads

For the 249 stocks in the data sample, true spreads were calculated using the LogNormal regression results of table 2 and compared to the censored spreads of each stock. Censored spreads were calculated from the underlying continuous true spreads using equation 16.

\[ \delta_{\text{true}} = 10^{4.443 - 0.420 \log T_s + 0.493 \log \sigma_s} \]

The comparison between the true spreads and censored spreads is graphed in figure 4, the difference is the excess spread due to tick size explicitly introduced by the tick size censoring function (equation 16). From the graph it is readily seen that where censored spreads are well above the minimum spread line, then there is little or no difference between true spreads and censored spreads (no excess spread). However for lower priced stock where the censored spread is near or on the minimum spread line there is significant difference between true and censored spreads.

Figure 4: True spreads compared to censored spreads.
3.6 Censored Spreads as an Estimate of Observed Spreads

The model of censored spreads developed in this paper should be able to predict observed spreads accurately and without bias. This is readily tested by modelling observed spreads as a linear regression of censored spreads. So the censored spreads are calculated using the results from the LogNormal continuous spread model and fitted to the following linear regression:

$$S_{\text{observed}} = k S_{\text{censored}} + b + \varepsilon_s$$  \hspace{1cm} (17)

The intercept term $b$ in equation 17 is not significant so the regression is refitted without an intercept term and this regression shows that $k$ (slope) is not significantly different from 1. The $R^2$ of the regression is 0.86, so censored spreads are an accurate and unbiased estimate of observed spreads.

![Comparison of Observed and Censored Spreads](image)

Figure 5: The Relationship between Censored Spreads and Observed Spreads (249 stocks, slope 0.989, slope std error 0.021, no significant intercept)
4 Summary

The loglikelihood spread regression model suggested by Harris [8] [9] is generalized and applied to effective spread data from the Australian Stock Exchange. This model enables the identification of an unobserved continuous spread distribution related to the cost of the supply of liquidity. This unobserved continuous spread distribution is partitioned (‘censored’) by market tick size into the observed discrete spreads.

For some higher turnover, low priced stocks traded on the ASX, the 1 cent minimum tick size is large and all spreads are observed at the minimum 0.5 cent spread. This implies that ordinary least squares linear regression models of spreads on the ASX are miss-specified because this tick size censoring is not addressed by the regression. The loglikelihood spread regression model overcomes this problem and allows a model of unobserved continuous true spreads to be developed.

The continuous maximum likelihood regression model is robust. Continuous spread regression models were developed with 4 different parameterized distributions (see table 2) with similar regression results for turnover and volatility, all rejected price as significant. Stock turnover is a proxy for the scarcity of liquidity and increasing turnover results in decreasing spread (Demsetz [6]). Volatility is a determinant of spread because of the extra risk taken by a limit order trader when providing liquidity for a volatile stock (Copeland and Galai [5]). Price is not an important determinant of the unobserved continuous true spread and this is intuitively appealing - spread costs of trading of stock should not be influenced by nominal stock price. For example, in negotiating the cost of supplying $1m of stock, liquidity providers and demanders are neutral as whether that stock liquidity is supplied as $1 \times 1,000,000$ stock units or $100 \times 10,000$ stock units.

Not only does the continuous spread regression allow the investigation of the underlying influences on spreads but it also gives us a powerful tool for investigating the effect of exchange tick size policy on the cost of trading on pure limit order markets.
References


