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**Increasing and Decreasing Annuities and Time
Reversal
by
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Increasing and Decreasing Annuities and Time Reversal

by

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Abstract

Standard formulae for decreasing annuities are derived from those for increasing annuities by reversing the direction of the time axis.

Keywords: Decreasing annuities, increasing annuities, time reversal.

1. Introduction

The standard formulae for decreasing annuities can all be derived from first principles. They can also be derived from the formulae for increasing annuities by using the fact that the sum of corresponding increasing and decreasing annuities produces some multiple of a level annuity. For example:

$$(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} = (n+1)a_{\overline{n}|} \quad (1.1)$$

$$(I\bar{a})_{\overline{n}|} + (D\bar{a})_{\overline{n}|} = (n+1)\bar{a}_{\overline{n}|} \quad (1.2)$$

$$(\bar{I}\bar{a})_{\overline{n}|} + (\bar{D}\bar{a})_{\overline{n}|} = n\bar{a}_{\overline{n}|} \quad (1.3)$$

These methods of derivation can be found in many mathematics of finance textbooks, such as Broverman (1996).

If time-lines showing the payment streams of an increasing annuity and its corresponding decreasing annuity are examined, the obvious major difference between the diagrams is that for the increasing annuity the payments increase as time moves forward while for the decreasing annuity the payments increase as time moves backwards. This suggests that it may also be possible to derive the formulae for decreasing annuities from those for increasing annuities by reversing the direction of the time axis. This paper shows how this may be done.

2. Basic compound interest functions

I will use the normal compound interest symbols when the time axis runs in the usual direction and starred symbols to denote the corresponding quantities when the time axis has been reversed.

If the axis is reversed, then the value of an investment declines as time increases. Accumulating with compound interest for t years on the reversed time axis is equivalent to discounting for t years on the normal time axis. Hence:

$$1 + i^* = v \quad (2.1)$$

Rearranging (2.1) gives

$$i^* = v - 1 = -d \quad (2.2)$$

Note that in the usual situation where i is positive, i^* will be negative. Inverting (2.1) gives

$$v^* = 1 + i \quad (2.3)$$

Then

$$d^* = 1 - v^* = -i \quad (2.4)$$

Taking the logarithm of (2.1) gives

$$\delta^* = -\delta \quad (2.5)$$

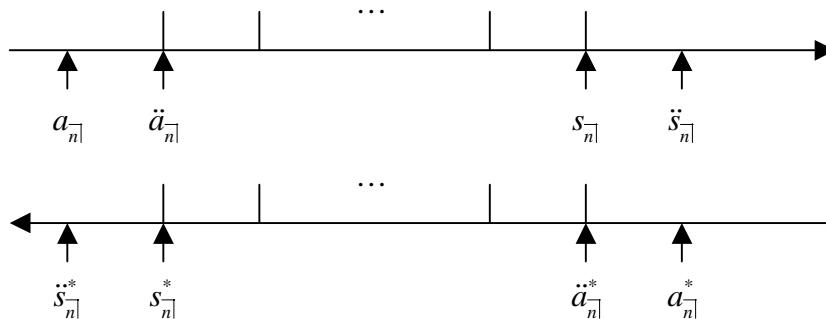
3. Level Annuities

In this section we derive some time reversal relationships involving level annuities. It is not being suggested that these relationships are an efficient way of deriving formulae for level annuities. These relationships are developed solely because they become useful in the next section when we derive formulae for decreasing annuities.

For ease of explanation, the following derivations will consider payments occurring annually and assume we are dealing with effective annual interest rates. However, the methods will work for any arbitrary time period desired.

In the following diagram the first time-line shows payments of 1 pa for n years. The positions of the standard discrete annuities are indicated.

In the second time-line, the payments have not been moved but the direction of the time axis has been reversed. That is, money still increases in value as it moves towards the right, but this direction is now regarded as moving backwards in time. The annuity symbols have again been placed in their correct positions on the diagram, this time with starred symbols. For example, s_n^* gives the value of the payments as at the date of the last payment, which is now the left-most payment since the time axis is reversed.



Comparing the two diagrams gives the following relationships.

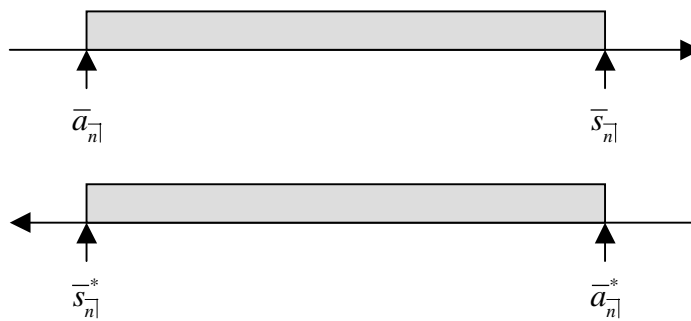
$$a_n^* = \ddot{s}_n \quad (3.1)$$

$$\ddot{a}_n^* = s_n \quad (3.2)$$

$$s_n^* = \ddot{a}_n \quad (3.3)$$

$$\ddot{s}_n^* = a_n \quad (3.4)$$

Here are the corresponding diagrams for continuous level annuities. The shaded bar indicates continuous payments at the rate of 1 pa for n years.



These diagrams give the following relationships.

$$\bar{a}_{n|}^* = \bar{s}_{n|} \quad (3.5)$$

$$\bar{s}_{n|}^* = \bar{a}_{n|} \quad (3.6)$$

If you find these equations unconvincing, you can always verify them using the appropriate equations selected from (2.1) to (2.5). For example, equation (3.1) can be verified as follows.

$$a_{n|}^* = \frac{1 - v^{*n}}{i^*} = \frac{1 - (1+i)^n}{-d} = \frac{(1+i)^n - 1}{d} = \ddot{s}_{n|}$$

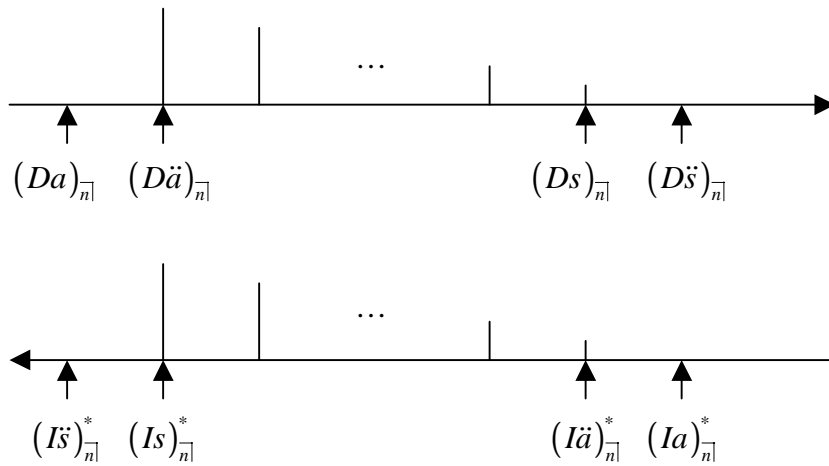
However, I would not want to encourage that this form of verification be done routinely since it would mean that the time reversal process for deriving decreasing annuity formulae would be grossly inefficient compared to the techniques given in section 1.

4. Decreasing Annuities

In this section we are assuming that we have derived the formulae for increasing annuities from first principles and now wish to derive those for decreasing annuities.

The following diagram shows payments of $n, n-1, n-2, \dots, 1$ spaced one year apart. The positions of the standard discrete decreasing annuities are indicated. (In all the remaining diagrams, the vertical scale has been compressed relative to the horizontal scale to save space.)

As before, the second time-line does not move the payments but the direction of the time axis has been reversed. Note that on this time-line as we move left time is increasing and the size of the payments increases. That is, this time-line will show us increasing annuities. The increasing annuity symbols have been placed in their correct positions on the diagram, with starred symbols being used due to the time axis having been reversed.



This diagram gives the following relationships.

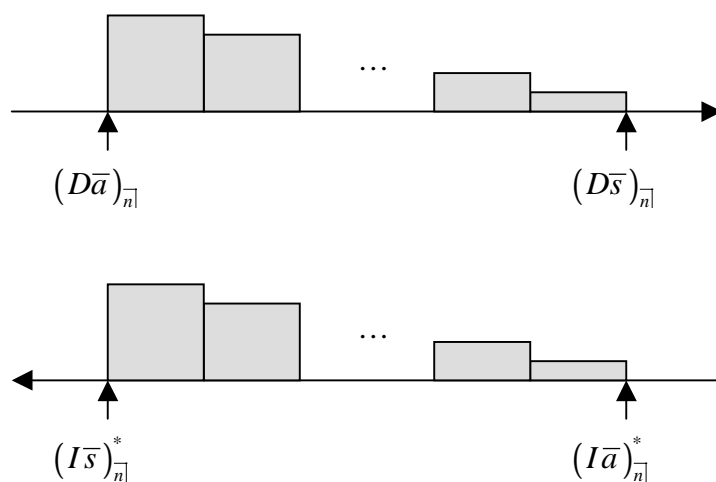
$$(Da)_{\overline{n}|} = (I\ddot{s})_{\overline{n}|}^* \tag{4.1}$$

$$(D\ddot{a})_{\overline{n}|} = (Is)_{\overline{n}|}^* \tag{4.2}$$

$$(Ds)_{\overline{n}|} = (I\ddot{a})_{\overline{n}|}^* \tag{4.3}$$

$$(D\ddot{s})_{\overline{n}|} = (Ia)_{\overline{n}|}^* \tag{4.4}$$

Here are the corresponding diagrams for the step decreasing continuous annuities. The shaded regions indicate payments at the rate of n pa for the first year, $n-1$ pa for the second year, and so on for n years.

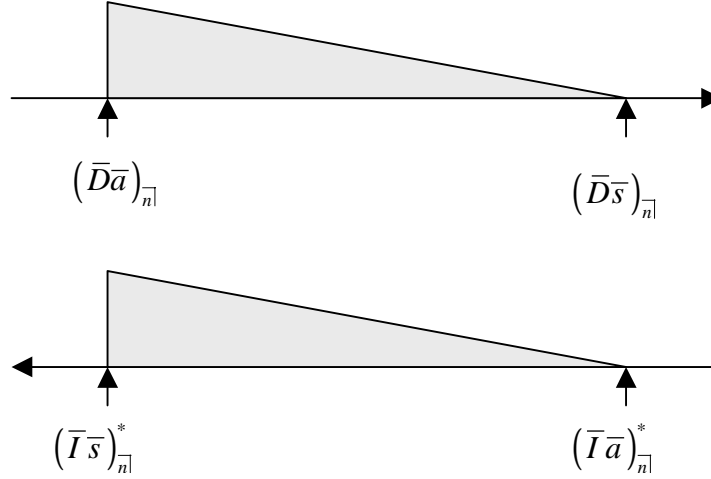


This diagram gives the following relationships.

$$(D\bar{a})_{\overline{n}|} = (I\bar{s})_{\overline{n}|}^* \quad (4.5)$$

$$(D\bar{s})_{\overline{n}|} = (I\bar{a})_{\overline{n}|}^* \quad (4.6)$$

Here are the diagrams for the continuously decreasing continuous annuities.



This diagram gives the final two required formulae.

$$(\bar{D}\bar{a})_{\overline{n}|} = (\bar{I}\bar{s})_{\overline{n}|}^* \quad (4.7)$$

$$(\bar{D}\bar{s})_{\overline{n}|} = (\bar{I}\bar{a})_{\overline{n}|}^* \quad (4.8)$$

All the conceptually difficult work is now over; all that remains is the algebra. We develop equations (4.1) to (4.8), substituting in where necessary from (2.1) to (2.5) and from (3.1) to (3.6). Here are the results.

$$(Da)_{\overline{n}|} = (I\dot{s})_{\overline{n}|}^* = \frac{\ddot{s}_{\overline{n}|}^* - n}{d^*} = \frac{a_{\overline{n}|} - n}{-i} = \frac{n - a_{\overline{n}|}}{i}$$

$$(D\ddot{a})_{\overline{n}|} = (I\dot{s})_{\overline{n}|}^* = \frac{\ddot{s}_{\overline{n}|}^* - n}{i^*} = \frac{a_{\overline{n}|} - n}{-d} = \frac{n - a_{\overline{n}|}}{d}$$

$$(Ds)_{\overline{n}|} = (I\ddot{a})_{\overline{n}|}^* = \frac{\ddot{a}_{\overline{n}|}^* - nv^{*n}}{d^*} = \frac{s_{\overline{n}|} - n(1+i)^n}{-i} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

$$(D\ddot{s})_{\overline{n}|} = (I\ddot{a})_{\overline{n}|}^* = \frac{\ddot{a}_{\overline{n}|}^* - nv^{*n}}{i^*} = \frac{s_{\overline{n}|} - n(1+i)^n}{-d} = \frac{n(1+i)^n - s_{\overline{n}|}}{d}$$

$$(D\bar{a})_{\overline{n}|} = (I\bar{s})_{\overline{n}|}^* = \frac{\ddot{s}_{\overline{n}|}^* - n}{\delta^*} = \frac{a_{\overline{n}|} - n}{-\delta} = \frac{n - a_{\overline{n}|}}{\delta}$$

$$\begin{aligned}
(D\bar{s})_{\overline{n}|} &= (I\bar{a})_{\overline{n}|}^* = \frac{\ddot{a}_{\overline{n}|}^* - nv^{*n}}{\delta^*} = \frac{s_{\overline{n}|} - n(1+i)^n}{-\delta} = \frac{n(1+i)^n - s_{\overline{n}|}}{\delta} \\
(\bar{D}\bar{a})_{\overline{n}|} &= (\bar{I}\bar{s})_{\overline{n}|}^* = \frac{\bar{s}_{\overline{n}|}^* - n}{\delta^*} = \frac{\bar{a}_{\overline{n}|} - n}{-\delta} = \frac{n - \bar{a}_{\overline{n}|}}{\delta} \\
(\bar{D}\bar{s})_{\overline{n}|} &= (\bar{I}\bar{a})_{\overline{n}|}^* = \frac{\bar{a}_{\overline{n}|}^* - nv^{*n}}{\delta^*} = \frac{\bar{s}_{\overline{n}|} - n(1+i)^n}{-\delta} = \frac{n(1+i)^n - \bar{s}_{\overline{n}|}}{\delta}
\end{aligned}$$

5. Efficiency of the Derivation

In practice, we would not derive all 8 decreasing annuity formulae using time reversal. This was done above merely to demonstrate that the process does work in all 8 cases.

When deriving the formulae for increasing annuities, we would usually only derive those for $(Ia)_{\overline{n}|}$ and $(\bar{I}\bar{a})_{\overline{n}|}$ from first principles. All the other formulae can be derived from these using appropriate accumulation terms. For example:

$$\begin{aligned}
(Is)_{\overline{n}|} &= (Ia)_{\overline{n}|} \times (1+i)^n \\
(I\ddot{a})_{\overline{n}|} &= (Ia)_{\overline{n}|} \times (1+i) \\
(I\bar{a})_{\overline{n}|} &= (Ia)_{\overline{n}|} \times \frac{i}{\delta}
\end{aligned}$$

Similar comments apply to decreasing annuities. We really only need use the time reversal approach twice, to determine the formulae for $(Da)_{\overline{n}|}$ and $(\bar{D}\bar{a})_{\overline{n}|}$. All other decreasing annuity formulae can then be derived using appropriate accumulation terms.

The relevant question to ask here is whether the time reversal derivation is more efficient than the more traditional methods outlined briefly in the introduction. The answer to this is probably subjective and will depend on what types of derivation the reader feels most comfortable with. It seems to me that the algebra involved in the time reversal method is slightly simpler than the traditional method. However, the time reversal method is conceptually considerably more difficult to understand; reversing the time axis is not a concept that comes naturally. (Perhaps it comes more naturally to teachers who are used to mentally reversing the horizontal axis of any graph so that they can face their audience and point in the directions that match the graph being projected onto the wall behind them!) Hence I do not expect the time reversal technique to ever supplant the traditional derivation. Still, I hope that at least some readers will share my view that it is an elegant piece of mathematics.

Bibliography

Broverman, Samuel A. 1996. Mathematics of Investment and Credit. Second Edition. Actex Publications Inc.