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Expanding the Geometric Mean Return

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EXPANDING THE GEOMETRIC MEAN RETURN

BY

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ABSTRACT

The generally accepted definition of the geometric mean of a set of returns is expanded using Taylor series expansions. The first term in the expansion is the arithmetic mean return. The second term in the expansion is minus half of the biased sample variance of the returns. In this paper the next six terms in the expansion are also found. The poor convergence when large and varied returns are involved is highlighted by an example.

KEYWORDS

Arithmetic mean, returns, Taylor series expansions

1. INTRODUCTION

Two different methods of expanding the geometric mean return are investigated in this paper. Both methods produce identical results. While the first two terms in the expansion are relatively simple expressions, higher order terms are more complex.

Employing many of the terms in the expansion does not necessarily ensure rapid convergence towards the geometric mean. As the order of the terms increases, poor convergence towards zero is quite possible, as is demonstrated by an example.

The methods used in this paper are purely mathematical, as they are based only upon Taylor series expansions. Therefore the results of this paper are expected to hold true wherever geometric returns are used, regardless of whether or not modern financial theory or any other theory applies.

2. DERIVING THE TERMS IN THE EXPANSION

Consider a set of n returns r_i for $i = 1, 2, 3, \dots, n$. The definition of the geometric mean of the returns is \bar{r}_g where

$$\bar{r}_g = \left[\prod_{i=1}^n (1 + r_i) \right]^{\frac{1}{n}} - 1 .$$

Let us define $x = \ln(1 + \bar{r}_g)$. It is worth noting that x is equal to the mean log return:

$$x = \frac{1}{n} \sum_{i=1}^n \ln(1 + r_i) .$$

Using Taylor series expansions yields

$$\begin{aligned}
x &= \frac{1}{n} \sum_{i=1}^n \left(r_i - \frac{r_i^2}{2} + \frac{r_i^3}{3} - \frac{r_i^4}{4} + \dots \right) \\
&= \frac{\sum_{i=1}^n r_i}{n} - \frac{1}{2} \times \frac{\sum_{i=1}^n r_i^2}{n} + \frac{1}{3} \times \frac{\sum_{i=1}^n r_i^3}{n} - \frac{1}{4} \times \frac{\sum_{i=1}^n r_i^4}{n} + \dots \\
\therefore x &= \bar{r} - \frac{\overline{r^2}}{2} + \frac{\overline{r^3}}{3} - \frac{\overline{r^4}}{4} + \dots
\end{aligned} \tag{1}$$

Furthermore note that $\bar{r}_g = e^x - 1$, and from Taylor series expansions,

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \tag{2}$$

Substituting equation (1) into equation (2) leads to the complete expansion of the geometric mean return \bar{r}_g . For notational simplicity, define T_k as the k^{th} order term in the expansion of the geometric mean return. Working through the substitution yields

$$T_1 = \bar{r} .$$

Therefore the first term in the expansion of the geometric mean return is the arithmetic mean return. Furthermore,

$$T_2 = \frac{(\bar{r})^2}{2} - \frac{\overline{r^2}}{2} .$$

Therefore the second term in the expansion of the geometric mean return is minus half of the biased sample variance of the returns. Note that T_2 involves the biased sample variance (which is of the form s_n^2), rather than the unbiased sample variance (which is of the form s_{n-1}^2).

Persistence leads to more terms in the expansion of the geometric mean return:

$$\begin{aligned}
T_3 &= \frac{(\bar{r})^3}{6} - \frac{\overline{r^2} \bar{r}}{2} + \frac{\overline{r^3}}{3} , \\
T_4 &= \frac{(\bar{r})^4}{24} - \frac{\overline{r^2} (\bar{r})^2}{4} + \frac{\overline{r^3} \bar{r}}{3} + \frac{(\overline{r^2})^2}{8} - \frac{\overline{r^4}}{4} , \\
T_5 &= \frac{(\bar{r})^5}{120} - \frac{\overline{r^2} (\bar{r})^3}{12} + \frac{\overline{r^3} (\bar{r})^2}{6} - \frac{\overline{r^4} \bar{r}}{4} + \frac{(\overline{r^2})^2 \bar{r}}{8} - \frac{\overline{r^3} \overline{r^2}}{6} + \frac{\overline{r^5}}{5} ,
\end{aligned}$$

$$\begin{aligned}
T_6 &= \frac{(\bar{r})^6}{720} - \frac{\bar{r}^2(\bar{r})^4}{48} + \frac{\bar{r}^3(\bar{r})^3}{18} + \frac{(\bar{r}^2)^2(\bar{r})^2}{16} - \frac{\bar{r}^3\bar{r}^2\bar{r}}{6} - \frac{(\bar{r}^2)^3}{48} - \frac{\bar{r}^4(\bar{r})^2}{8} + \frac{\bar{r}^5\bar{r}}{5} + \frac{(\bar{r}^3)^2}{18} + \frac{\bar{r}^4\bar{r}^2}{8} - \frac{\bar{r}^6}{6} , \\
T_7 &= \frac{(\bar{r})^7}{5040} - \frac{\bar{r}^2(\bar{r})^5}{240} + \frac{\bar{r}^3(\bar{r})^4}{72} + \frac{(\bar{r}^2)^2(\bar{r})^3}{48} - \frac{\bar{r}^4(\bar{r})^3}{24} - \frac{\bar{r}^3\bar{r}^2(\bar{r})^2}{12} - \frac{(\bar{r}^2)^3\bar{r}}{48} + \frac{\bar{r}^5(\bar{r})^2}{10} + \frac{\bar{r}^4\bar{r}^2\bar{r}}{8} \\
&\quad + \frac{\bar{r}^3(\bar{r}^2)^2}{24} + \frac{(\bar{r}^3)^2\bar{r}}{18} - \frac{\bar{r}^6\bar{r}}{6} - \frac{\bar{r}^5\bar{r}^2}{10} - \frac{\bar{r}^4\bar{r}^3}{12} + \frac{\bar{r}^7}{7} , \\
T_8 &= \frac{(\bar{r})^8}{40320} - \frac{\bar{r}^2(\bar{r})^6}{1440} + \frac{\bar{r}^3(\bar{r})^5}{360} + \frac{(\bar{r}^2)^2(\bar{r})^4}{192} - \frac{\bar{r}^4(\bar{r})^4}{96} - \frac{\bar{r}^3\bar{r}^2(\bar{r})^3}{36} - \frac{(\bar{r}^2)^3(\bar{r})^2}{96} + \frac{\bar{r}^5(\bar{r})^3}{30} + \frac{\bar{r}^4\bar{r}^2(\bar{r})^2}{16} \\
&\quad + \frac{\bar{r}^3(\bar{r}^2)^2\bar{r}}{24} + \frac{(\bar{r}^2)^4}{384} + \frac{(\bar{r}^3)^2(\bar{r})^2}{36} - \frac{\bar{r}^6(\bar{r})^2}{12} - \frac{\bar{r}^5\bar{r}^2\bar{r}}{10} - \frac{\bar{r}^4(\bar{r}^2)^2}{32} - \frac{(\bar{r}^3)^2\bar{r}^2}{36} - \frac{\bar{r}^4\bar{r}^3\bar{r}}{12} + \frac{\bar{r}^7\bar{r}}{7} + \frac{\bar{r}^6\bar{r}^2}{12} \\
&\quad + \frac{\bar{r}^5\bar{r}^3}{15} + \frac{(\bar{r}^4)^2}{32} - \frac{\bar{r}^8}{8} .
\end{aligned}$$

Higher order terms can be found by continuing the expansions to the desired order. However as the order increases, the number of sub-terms in each expression increases much more rapidly, decreasing the useability of each expression. For this reason only terms up to an order of eight have been considered.

3. A DERIVATION BASED ON BINOMIAL EXPANSIONS

As an alternative to the above derivation based on log returns, the geometric mean return can also be expanded using a binomial expansion as follows:

$$\begin{aligned}
\bar{r}_g &= \left[\prod_{i=1}^n (1 + r_i) \right]^{\frac{1}{n}} - 1 = \prod_{i=1}^n (1 + r_i)^{\frac{1}{n}} - 1 . \\
\therefore \bar{r}_g &= \prod_{i=1}^n \left[1 + \frac{1}{n} \frac{r_i}{1!} + \frac{1}{n} \left(\frac{1}{n} - 1 \right) \frac{r_i^2}{2!} + \frac{1}{n} \left(\frac{1}{n} - 1 \right) \left(\frac{1}{n} - 2 \right) \frac{r_i^3}{3!} + \dots \right] - 1 . \quad (3)
\end{aligned}$$

Some consideration of equation (3) will show that terms with a given subscript are never multiplied by any other term of that same subscript. For example, a term in r_1 is never multiplied by a term in r_1^2 . However all other combinations are possible – for example, each term with a subscript of 1 will be multiplied by all terms with a subscript of (say) 4.

Thus the expansion of equation (3) produces many series of terms. Each series can be regarded as a summation over the relevant subscripts, except that no two subscripts in each summand can be the same. These partial summations are evident in the second and third order terms that result from the expansion of equation (3):

$$\begin{aligned} \bar{r}_g = & \left\{ \frac{1}{n} \sum_{i=1}^n r_i \right\} + \left\{ \frac{1}{2} \times \frac{1}{n} \frac{1}{n} \left(\sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n r_i r_j \right) + \frac{1}{n} \left(\frac{1}{n} - 1 \right) \frac{1}{2!} \sum_{i=1}^n r_i^2 \right\} \\ & + \left\{ \frac{1}{6} \times \frac{1}{n} \frac{1}{n} \frac{1}{n} \left(\sum_{\substack{i=1 \\ i \neq j, i \neq k, j \neq k}}^n \sum_{j=1}^n \sum_{k=1}^n r_i r_j r_k \right) + \frac{1}{n} \left(\frac{1}{n} - 1 \right) \frac{1}{2!} \frac{1}{n} \left(\sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n r_i^2 r_j \right) + \frac{1}{n} \left(\frac{1}{n} - 1 \right) \left(\frac{1}{n} - 2 \right) \frac{1}{3!} \sum_{i=1}^n r_i^3 \right\} + \dots \end{aligned}$$

Each partial summation can be expressed in terms of complete summations. For example, the following is required to simplify the second order term:

$$\sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n r_i r_j = \left(\sum_{i=1}^n r_i \right)^2 - \sum_{i=1}^n r_i^2 .$$

Finding such formulae for all the partial summations, substituting into the expansion of equation (3) and then simplifying is algebraically tedious. However this process eventually produces exactly the same expressions as previously found for T_1 through to T_8 .

4. A SEMI-INDEPENDENT CHECK

When all returns are equal, then by definition the geometric mean return equals the arithmetic mean return.

Also when all returns are equal, then for any integer a ,

$$\overline{r^a} = (\bar{r})^a .$$

Substituting the above result into T_2 through to T_8 yields zero for each of these T terms. Therefore when all returns are equal, only the first term in the expansion of the geometric mean, T_1 , is required to give the exact answer.

This result represents a significant self-check on the correctness of the formulae, and can also be expected to apply when T terms of an order higher than eight are considered.

5. CONVERGENCE

When returns are relatively small, the terms rapidly converge to zero as the order of the terms increases. For example, consider the returns 5.3%, 2.5%, -1.2%, 3.6%, -3.7% and 4.2%. In this case the terms are well behaved as is shown in Table 1, and there is excellent agreement between the sum of the terms and the geometric mean return.

Table 1: Convergence with small returns

| Term | Value (%) |
|-------------|-----------------------|
| T_1 | 1.7833333333% |
| T_2 | -0.0508236111% |
| T_3 | 0.0001982902% |
| T_4 | -0.0000272206% |
| T_5 | 0.0000003975% |
| T_6 | -0.0000000338% |
| T_7 | 0.0000000009% |
| T_8 | <u>-0.0000000001%</u> |
| | 1.7326811564% |
| \bar{r}_g | 1.7326811564% |

However if any of the returns are large and varied, or if there are relatively few returns (that is, n is small), then convergence is quite poor. Consider now just two returns, +100% and -40%. As shown in Table 2, incorporating all terms up to the 8th order term still leaves a significant discrepancy of 0.72% between the sum of the terms and the geometric mean return.

Table 2: Convergence with large and varied returns

| Term | Value (%) |
|-------------|-----------------------|
| T_1 | 30.0000000000% |
| T_2 | -24.5000000000% |
| T_3 | 7.3500000000% |
| T_4 | -5.2062500000% |
| T_5 | 3.3626250000% |
| T_6 | -2.5544312500% |
| T_7 | 1.9728318750% |
| T_8 | <u>-1.6003633516%</u> |
| | 8.8244122734% |
| \bar{r}_g | 9.5445115010% |

This shows that accuracy cannot be blindly assumed even when so many of the most significant terms are taken into account.

6. CONCLUSIONS

In this paper the first eight terms in the expansion of the geometric mean return have been determined. Higher order terms can also be found by extending the process described in this paper. Although using the eight most significant terms in the expansion of the geometric mean may seem to imply a high level of accuracy, large inaccuracies are possible if there are large and varied returns involved, or if the number of returns is low.