A BSDE Approach to a Risk-Based Optimal Investment of an Insurer

Robert J. Elliott *   Tak Kuen Siu †

*Haskayne School of Business, University of Calgary, CANADA; School of Mathematical Sciences, University of Adelaide, AUSTRALIA

†Department of Applied Finance and Actuarial Studies, Faculty of Business and Economics, Macquarie University, Sydney, AUSTRALIA

§1. Optimal Investment of Insurers: Background and Literature

- Asset liability management for insurance businesses

- Traditional investment assets: Fixed interest securities

- Nowadays, investment items include risky assets like equities, share indices, derivatives, and many others.

- Quantitative Models: Make rational, scientific and justified investment decisions

- Optimal asset allocation in financial economics: Markowitz (1952), Samuelson (1969) and Merton (1971)

- Insurance risk attributed to liabilities or claims is not included.

- Classic works on modeling insurance risk: Borch (1967, 1969), Bühlmann (1970) and Gerber (1972).

- Stochastic Models for both insurance and financial risks are required.

- Some literature on optimal investment of insurance companies: Browne (1995, 1997, 1999), Hipp and Taksar (2000), Hipp and Plum (2000), Liu and Yang (2004) and Yang and Zhang (2005)

- Insurance Risk Processes: Compound Poisson processes, diffusion approximations, jump-diffusion processes

- Objective functions: Minimize ruin probability, maximizing expected utility

- Key method: HJB dynamic programming approach

Risk measures in finance

- Value at Risk (VaR): popular, but not sub-additive and time-consistent, as well as leads to some bizarre and sub-optimal decisions if used as a binding constraint in portfolio selection, (Basak and Shapiro (2001))

- Coherent risk measures by Artzner et al. (1999): a remedy for some defects of VaR, but cannot incorporate liquidity risk of a large trading position

- Convex risk measures by Frittelli and Rosazza-Gianin (2002) and Föllmer Schied (2002): Generalization of coherent risk measures by incorporating liquidity risk, a key issue as highlighted by the recent global financial crisis

Keypoints of our work

- Discuss a risk-based, optimal investment problem for an insurer using a BSDE approach

- Use a convex risk measure as the objective function

- Formulate the problem as a two-player, zero-sum, stochastic differential game

- Solve the game problem using the BSDE approach

- Closed-form solutions to the optimal strategies

§2. The Model Dynamics

- Consider a continuous-time economy with two investment assets, namely, a bond B and a share S

- The price process of B evolves over time as:

$$B(t) = \exp\left(\int_0^t r(u)du\right) , \quad B(0) = 1 ,$$

where $r(t)$ is the risk-free interest rate at time t

- The price process of S is governed by a GBM:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW_1(t) ,$$

where $\{W_1(t)\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$.

- For each $t \in [0, T]$, let $N(t)$ be the number of claims in the time interval $[0, t]$ and Y_i be the size of the i^{th} individual claim, where $i = 1, 2, \cdots$.

- We suppose that under P:

  1. $\{N(t)|t \in \mathcal{T}\}$ is a Poisson process with a constant intensity λ;

  2. $\{Y_i|i = 1, 2, \cdots\}$ are independent and identically distributed, (i.i.d.), nonnegative random variables with a common continuous distribution function $F(y)$ having the first and second moments μ_Y and σ_Y^2, respectively, where $\mu_Y, \sigma_Y < \infty$;

  3. $\{N(t)|t \in \mathcal{T}\}$, $\{W_1(t)|t \in \mathcal{T}\}$ and $\{Y_i|i = 1, 2, \cdots\}$ are independent.

- The aggregate claims amount up to time t is:

$$Z(t) = \sum_{i=1}^{N(t)} Y_i \ .$$

- The premium rate associated with the relative security loading κ:

$$c(\kappa) = (1+\kappa)\lambda\mu_Y \ ,$$

where $\kappa > 0$.

- The classical Cramer-Lundberg model for the surplus process without investment:

$$U^\kappa(t) = c(\kappa)t - Z(t) \ , \quad U^\kappa(0) = u_0 \ .$$

- Without loss of generality, we suppose $\lambda = 1$.

- **Theorem 1 (Grandell, 1991):** *Let* $\{W^2(t)\}$ *be a second standard Brownian motion on* $(\Omega, \mathcal{F}, P)$*. Write* $\mathcal{D}[0,T]$ *for the space of càdlàg functions on* $[0,T]$ *endowed with the Skorohod topology. Define the diffusion process* $\{R(t)\}$ *by:*

$$R(t) = u_0 + \mu_Y t + \sigma_Y W_2(t) \ .$$

Then

$$\{\kappa U^\kappa(t/\kappa^2)\} \to \{R(t)\} \ ,$$

in $\mathcal{D}[0,T]$ *as* $\kappa \to 0$*, where the convergence is in the sense of Skorohod topology.*

- Assume that $Cov(W_1(t), W_2(t)) = \int_0^t \rho_{12}(u)du$.

Information Structure

- $\{\mathcal{F}^R(t)\}$: The P-completed, right-continuous, natural filtration generated by the insurance risk process

- $\{\mathcal{F}^S(t)\}$: The P-completed, right-continuous, natural filtration generated by the share price process

- $\mathcal{G}(t) = \mathcal{F}^R(t) \vee \mathcal{F}^S(t)$

- $\{\mathcal{G}(t)\}$: The observable flow of information

- $\pi(t)$: The amount of money the insurer invests in the share at time t

- Let $\boldsymbol{\sigma}(t, \pi(t)) := (\sigma(t)\pi(t), \sigma_Y)'$ and $\mathbf{W}(t) := (W_1(t), W_2(t))'$. Then the evolution of the surplus process of the insurer associated with the investment process $\{\pi(t)\}$ over time is:

$$\begin{aligned} dV(t) &= (\kappa\mu_Y + r(t)V(t) + \pi(t)(\mu(t) - r(t)))dt + \boldsymbol{\sigma}'(t, \pi(t))d\mathbf{W}(t) \ , \\ V(0) &= v_0 \ . \end{aligned}$$

- A portfolio process π is said to be admissible, (i.e. $\pi \in \mathcal{A}$), if it satisfies the following conditions:

  1. π is $\{\mathcal{G}(t)\}$-progressively measurable;

  2. $$\int_0^T [\pi(t)]^2 dt < \infty \ , \quad \mathcal{P}\text{-a.s.};$$

  3. the stochastic differential equation for the surplus process V has a unique strong solution;

  4. $$\int_0^T \Bigg(|\kappa\mu_Y + r(t)V(t) + \pi(t)(\mu(t) - r(t))| + ||\boldsymbol{\sigma}'(t, \pi(t))|| + ||\boldsymbol{\sigma}'(t, \pi(t))||^2 \Bigg) dt < \infty \ , P\text{-a.s.}$$

§3. Risk-based Optimal Investment for an Insurer

- **Definition 1** *Let $\mathcal{S}$ be the space of all lower-bounded random variables on the measurable space $(\Omega, \mathcal{G}(T))$. A convex risk measure ρ is a functional $\rho : \mathcal{S} \to \Re$ which satisfies the following three properties:*

  1. *If $X \in \mathcal{S}$ and $K \in \Re$, then*

  $$\rho(X + K) = \rho(L) - K \ .$$

  2. *For any $X_1, X_2 \in \mathcal{S}$, if $X_1(\omega) \leq X_2(\omega)$, for all $\omega \in \Omega$, then $\rho(X_1) \geq \rho(X_2)$.*

  3. *For any $X_1, X_2 \in \mathcal{S}$ and $\zeta \in (0, 1)$,*

  $$\rho(\zeta X_1 + (1 - \zeta) X_2) \leq \zeta \rho(X_1) + (1 - \zeta) \rho(X_2) \ .$$

- **Theorem 2 (Föllmer and Schied, 2002, and Frittelli and Rosazza-Gianin, 2002):** *Let $\mathcal{M}_a$ be a family of probability measures Q which are absolutely continuous with respect to P. Define a function $\eta : \mathcal{M}_a \to \Re$ such that $\eta(Q) < \infty$, $\forall Q \in \mathcal{M}_a$. Then for any convex risk measure $\rho(X)$ of $X \in \mathcal{S}$, there exists a family $\mathcal{M}_a$ and a function η such that*

$$\rho(X) = \sup_{Q \in \mathcal{M}_a} \{E_Q[-X] - \eta(Q)\} .$$

Here E_Q represents expectation under Q.

- We must specify the family of probability scenarios $\mathcal{M}_a$ and the penalty function η.

- Generate a family $\mathcal{M}_a$ by a set of probability measures of Girsanov transformation type

- Let $\{\boldsymbol{\theta}(t)\}$ be a $\{\mathcal{G}(t)\}$-predictable process, where $\boldsymbol{\theta}(t) := (\theta_1(t), \theta_2(t)) \in \Re^2$, such that

$$\int_0^T ||\boldsymbol{\theta}(t)||^2 dt < \infty \ , \quad P\text{-a.s.}$$

- Consider the $\{\mathcal{G}(t)\}$-adapted process $\{\Lambda^{\boldsymbol{\theta}}(t)\}$ defined by:

$$\Lambda^{\boldsymbol{\theta}}(t) := 1 + \int_0^t \Lambda^{\boldsymbol{\theta}}(u)\boldsymbol{\theta}'(u) d\mathbf{W}(u) \ .$$

- A process $\boldsymbol{\theta}$ is an admissible scenario if

  1. $\{\Lambda^{\boldsymbol{\theta}}(t)\}$ is a (G,P)-martingale;

  2. $$\int_0^T ||\boldsymbol{\theta}(t)||^2 dt < \infty \ , \quad \int_0^T ||\Lambda^{\boldsymbol{\theta}}(t)\boldsymbol{\theta}(t)||dt < \infty \ , \qquad P\text{-a.s.}$$

- Write $\boldsymbol{\Theta}$ for the space of admissible strategies $\boldsymbol{\theta}$.

- For each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, a new probability measure $Q_{\boldsymbol{\theta}} << P$ on $\mathcal{G}(T)$ is defined by putting:

$$\left.\frac{dQ_{\boldsymbol{\theta}}}{dP}\right|_{\mathcal{G}(T)} := \Lambda^{\boldsymbol{\theta}}(T) \ .$$

- Then we set

$$\mathcal{M}_a = \mathcal{M}_a(\boldsymbol{\Theta}) := \{Q_{\boldsymbol{\theta}} | \boldsymbol{\theta} \in \boldsymbol{\Theta}\} \ .$$

- Define, for each $(\pi, \boldsymbol{\theta}) \in \mathcal{A} \times \boldsymbol{\Theta}$, a controlled process $\{\mathbf{Y}(t)\}$, where

$$\begin{aligned} dY_1(t) &= Y_1(t)\boldsymbol{\theta}'(t)d\mathbf{W}(t) \ , \\ dY_2(t) &= (\kappa\mu_Y + r(t)Y_2(t) + \pi(t)(\mu(t) - r(t)))dt + \boldsymbol{\sigma}'(t, \pi(t))d\mathbf{W}(t) \ , \end{aligned}$$

  so that $\mathbf{Y}(t) = (Y_1(t), Y_2(t)) = (\Lambda^{\boldsymbol{\theta}}(t), V^{\pi}(t)) \in \Re^2$.

- Consider the following penalty function η:

$$\eta(Q_{\boldsymbol{\theta}}, \pi) := \mathsf{E}\left[\int_0^T \lambda(t, \mathbf{Y}(\cdot), \pi(t), \boldsymbol{\theta}(t))dt + h(\mathbf{Y}(T)) \right] \ ,$$
$$\forall (\pi, \boldsymbol{\theta}) \in \mathcal{A} \times \boldsymbol{\Theta} \ ,$$

  for some bounded, measurable convex functions λ and h.

- A convex risk measure can then be defined as:

$$\rho(Y_2(T)) := \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \{\mathsf{E}^{\boldsymbol{\theta}}[-Y_2(T)] - \eta(Q_{\boldsymbol{\theta}}, \pi)\} \ ,$$

where $\mathsf{E}^{\boldsymbol{\theta}}$ is expectation under $Q_{\boldsymbol{\theta}}$.

- The optimal investment problem of the insurer as a zero-sum stochastic differential game between the insurer and the market:

$$\Phi(y_2) := \inf_{\pi\in\mathcal{A}} \rho(Y_2(T)) = \inf_{\pi\in\mathcal{A}} \left\{ \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \{\mathsf{E}^{\boldsymbol{\theta}}[-Y_2(T)] - \eta(Q_{\boldsymbol{\theta}}, \pi)\} \right\} \ .$$

§4. The BSDE Solution to the Game Problem

- **Definition 2** *Let $\xi(T)$ be a real-valued, square-integrable, $\mathcal{G}(T)$-measurable terminal condition and $g : \Omega \times \mathcal{T} \times \Re \times \Re^2 \to \Re$ be a real-valued, $\{\mathcal{G}(t)\}$-progressively measurable, square-integrable, driver function. Then a solution of the BSDE associated with the driver function g and the terminal condition $\xi(T)$ is a pair of square-integrable, G-progressively measurable, processes $\{(X(t), \mathbf{Z}(t))\}$ such that*

$$X(t) \;=\; \xi(T) + \int_t^T g(u, X(u), \mathbf{Z}(u))du - \int_t^T \mathbf{Z}'(u)d\mathbf{W}(u) \ .$$

Or equivalently, in a differential form,

$$dX(t) = -g(t, X(t), \mathbf{Z}(t))dt + \mathbf{Z}'(t)d\mathbf{W}(t) \ , \quad X(T) = \xi(T) \ .$$

- **Theorem 3 (Existence and Uniqueness):** *Suppose the following conditions hold:*

  1. $\{g(\omega, t, 0, \mathbf{0})\}$ *is a real-valued, square-integrable,* $\{\mathcal{G}(t)\}$*-progressively measurable process;*

  2. *there exists a positive constant* K *such that* $\forall (x_1, x_2, \mathbf{z}_1, \mathbf{z}_2) \in \Re \times \Re \times \Re^2 \times \Re^2$,

     $$|g(\omega, t, x_1, \mathbf{z}_1) - g(\omega, t, x_2, \mathbf{z}_2)| \leq K(|x_1 - x_2| + ||\mathbf{z}_1 - \mathbf{z}_2||) \ ,$$
     $$(l \otimes P)\text{-a.e} \ ,$$

     where l *is the Lebesgue measure on* $\mathcal{T}$*.*

  Then the BSDE associated with the driver function g *and the terminal condition* $\xi(T)$ *has a unique solution.*

- **Theorem 4 (Comparison Theorem):** *Suppose*

  1. $(X_1, \mathbf{Z}_1)$ *and* $(X_2, \mathbf{Z}_2)$ *are the solutions of the two BSDEs associated with* $(g_1, \xi_1(T))$ *and* $(g_2, \xi_2(T))$*, respectively;*

  2. g_1 *and* g_2 *satisfy the two conditions in Theorem 3.*

  If $\xi_2(T) \leq \xi_1(T)$*,* P*-a.s., and* $\forall (x, \mathbf{z}) \in \Re \times \Re^2$*,* $g_2(\omega, t, x, \mathbf{z}) \geq g_1(\omega, t, x, \mathbf{z})$*,* $l \otimes P$*-a.e., then* $\forall (x, \mathbf{z}) \in \Re \times \Re^2$*,*

$$X_2(\omega, t) \leq X_1(\omega, t)\ , \quad \forall t \in \mathcal{T}\ , \quad P\text{-a.s.}$$

- Define a double-indexed process $\{\Gamma(t,u) | t \leq u\}$ as an adjoint process satisfying the following forward linear SDE:

$$d\Gamma(t,u) = \Gamma(t,u)(\beta(u)du + \boldsymbol{\gamma}'(u)d\mathbf{W}(u)) \ , \quad \Gamma(t,t) = 1 \ ,$$

so that $\Gamma(t,u)$ satisfies the following semi-group property:

$$\Gamma(t,s)\Gamma(s,u) = \Gamma(t,u) \ , \quad \forall t \leq s \leq u \ , P\text{-a.s.}$$

- **Theorem 5 (El Karoui et al., 1997)** *Let* $\{\beta(t)\}$ *and* $\{\gamma(t)\}$ *be bounded,* $\Re$*-valued and* $\Re^2$*-valued, respectively,* $\{\mathcal{G}(t)\}$*-progressively measurable processes. Suppose* $\{\alpha(t)\}$ *is a real-valued, square-integrable,* $\{\mathcal{G}(t)\}$*-progressively measurable process;* $\xi(T)$ *is a real-valued, square-integrable,* $\mathcal{G}(T)$*-measurable random variable. Consider the following linear BSDE:*

$$\begin{aligned} dX(t) &= -(\alpha(t) + \beta(t)X(t) + \gamma'(t)\mathbf{Z}(t))dt + \mathbf{Z}'(t)d\mathbf{W}(t) \ , \\ X(T) &= \xi(T) \ . \end{aligned}$$

Then this linear BSDE has a unique, square-integrable and $\{\mathcal{G}(t)\}$*-progressively measurable solution* $(X, \mathbf{Z})$*, and* X *has the following expectation representation:*

$$X(t) = E\Big[\xi(T)\Gamma(t,T) + \int_t^T \Gamma(t,u)\alpha(u)du | \mathcal{G}(t)\Big] \ .$$

- **Lemma 1** *Write*

$$\begin{aligned}
& \widetilde{\lambda}(t, \mathbf{Y}(\cdot), \pi(t), \boldsymbol{\theta}(t)) \\
:= \; & Y_1(t)(\kappa\mu_Y + r(t)Y_2(t) + \pi(t)(\mu(t) - r(t)) + \theta_1(t)\sigma(t)\pi(t) - \sigma_Y\theta_2(t) \\
& + \rho_{12}(t)\theta_2(t)\sigma(t)\pi(t) - \rho_{12}(t)\theta_1(t)\sigma_Y) + \lambda(t, \mathbf{Y}(\cdot), \pi(t), \boldsymbol{\theta}(t)) \ ,
\end{aligned}$$

and

$$\widetilde{J}^{\pi,\boldsymbol{\theta}}(\mathbf{y}) := E^{\mathbf{y}}\left[- \int_0^T \widetilde{\lambda}(t, \mathbf{Y}(\cdot), \pi(t), \boldsymbol{\theta}(t))dt - h(\mathbf{Y}(T)) \right] \ ,$$

where $E^{\mathbf{y}}$ *is the conditional expectation given* $\mathbf{Y}(0) = \mathbf{y}$ *under* P*.*

Then the stochastic differential game is equivalent to:

$$\widetilde{\Phi}(\mathbf{y}) = \widetilde{J}^{\pi^*,\boldsymbol{\theta}^*}(\mathbf{y}) = \inf_{\pi\in\mathcal{A}} \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \widetilde{J}^{\pi,\boldsymbol{\theta}}(\mathbf{y}) \ .$$

- Define the Hamiltonian H of the game problem by:

$$H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi(t), \boldsymbol{\theta}(t)) := -\widetilde{\lambda}(t, \mathbf{Y}(\cdot), \pi(t), \boldsymbol{\theta}(t)) \ .$$

- **Definition 3 (Isaacs' condition):** *The Hamiltonian H is said to satisfy Isaacs' condition if for a.s. all t and ω,*

$$\inf_{\pi \in \mathcal{A}} \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi(t), \boldsymbol{\theta}(t)) = \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \inf_{\pi \in \mathcal{A}} H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi(t), \boldsymbol{\theta}(t)) \ .$$

- **Lemma 2** *(***(Friedman, 1975, and Elliott, 1976):** *There exist two measurable, $\{\mathcal{G}(t)\}$-adapted, functions π^* and $\boldsymbol{\theta}^*$ such that*

$$\begin{aligned} &H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi^*(t, \mathbf{Y}(\cdot), \mathbf{z}), \boldsymbol{\theta}) \leq H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi^*(t, \mathbf{Y}(\cdot), \mathbf{z}), \boldsymbol{\theta}^*(t, \mathbf{Y}(\cdot), \mathbf{z})) \\ &\leq H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi, \boldsymbol{\theta}^*(t, \mathbf{Y}(\cdot), \mathbf{z})) \ , \quad \forall (\pi, \boldsymbol{\theta}) \in \mathcal{A} \times \boldsymbol{\Theta} \ , \end{aligned}$$

if and only if H satisfies Isaacs' condition.

- **Lemma 3** $H(t, \mathbf{y}(\cdot), \mathbf{z}, \pi^*(t, \mathbf{y}(\cdot), \mathbf{z}), \boldsymbol{\theta}^*(t, \mathbf{y}(\cdot), \mathbf{z}))$ *is Lipschitz in* $\mathbf{z}$*, uniformly in* $(t, \mathbf{y}(\cdot))$*.*

- **Theorem 6** *Suppose Isaacs' condition and the two conditions in Theorem 3 hold. Then there is a unique solution* $\{(X(t), \mathbf{Z}(t)) | t \in \mathcal{T}\}$ *of the BSDE associated with the driver function* $H(t, \mathbf{y}(\cdot), \mathbf{z}, \pi^*(t, \mathbf{y}(\cdot), \mathbf{z}), \boldsymbol{\theta}^*(t, \mathbf{y}(\cdot), \mathbf{z}))$ *and terminal condition* $h(\mathbf{Y}(T))$*:*

$$\begin{aligned} -dX(t) &= H(t, \mathbf{y}(\cdot), \mathbf{z}, \pi^*(t, \mathbf{y}(\cdot), \mathbf{z}), \boldsymbol{\theta}^*(t, \mathbf{y}(\cdot), \mathbf{z}))dt - \mathbf{Z}'(t)d\mathbf{W}(t) \ , \\ X(T) &= h(\mathbf{Y}(T)) \ . \end{aligned}$$

Further the pair of strategies $(\pi^*(t, \mathbf{y}(\cdot), \mathbf{z}), \boldsymbol{\theta}^*(t, \mathbf{y}(\cdot), \mathbf{z}))$ *is a saddle point of the zero-sum stochastic differential game, and*

$$X(0) = \widetilde{J}^{\pi^*, \boldsymbol{\theta}^*}(\mathbf{y}) = \inf_{\pi \in \mathcal{A}} \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \widetilde{J}^{\pi, \boldsymbol{\theta}}(\mathbf{y}) = \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \inf_{\pi \in \mathcal{A}} \widetilde{J}^{\pi, \boldsymbol{\theta}}(\mathbf{y}) \ .$$

- Suppose the control and the controlled state process are Markov.

- Write, for each $(t, \mathbf{y}) \in \mathcal{T} \times \Re^2$,

$$\begin{aligned}
\mathbf{\Sigma}(t, \mathbf{y}, \pi^*(t), \boldsymbol{\theta}^*(t)) &:= (y_1(\boldsymbol{\theta}^*(t))', \sigma'(\pi^*(t)))' \in \Re^2 \otimes \Re^2 \ , \\
\mathbf{A}(t, \mathbf{y}, \pi^*(t), \boldsymbol{\theta}^*(t)) &:= \mathbf{\Sigma}(t, \mathbf{y}, \pi^*(t), \boldsymbol{\theta}^*(t))\mathbf{\Sigma}'(t, \mathbf{y}, \pi^*(t), \boldsymbol{\theta}^*(t)) \in \Re^2 \otimes \Re^2 \ .
\end{aligned}$$

- Define the following second-order partial differential operator:

$$\mathcal{A}_{t,\mathbf{y}}(\pi^*, \boldsymbol{\theta}^*) := \sum_{i,j=1}^{n} a_{ij}(t, \mathbf{y}, \pi^*(t), \boldsymbol{\theta}^*(t))\partial^2_{y_i y_j} + (\kappa\mu_Y + r y_2 \pi^*(t)(\mu - r))\partial_{y_2} \ .$$

- We now relate the BSDE solution of the stochastic differential game to a classical solution of a PDE using the nonlinear Feynman-Kac formula

- **Theorem 7** *Let* $u(\cdot,\cdot) \in \mathcal{C}^{1,2}(\mathcal{T} \times \Re^2)$. *Suppose there is a constant* K *such that for each* $(t, \mathbf{y}) \in \mathcal{T} \times \Re^2$,

$$|u(t,\mathbf{y})| + ||\partial_{\mathbf{y}} u(t,\mathbf{y}) \mathbf{\Sigma}(t, \mathbf{y}, \pi^*(t), \boldsymbol{\theta}^*(t))|| \leq K(1 + ||\mathbf{y}||) \ .$$

Assume further that $u(t, \mathbf{y})$ *is the solution of the semi-linear parabolic PDE:*

$$\partial_t u(t,\mathbf{y}) + \mathcal{A}_{t,\mathbf{y}}(\pi^*, \boldsymbol{\theta}^*)[u(t,\mathbf{y})] - \widetilde{\lambda}(t, \mathbf{y}(\cdot), \pi^*(t), \boldsymbol{\theta}^*(t)) = 0 \ ,$$
$$u(T, \mathbf{y}) = h(\mathbf{y}) \ .$$

If, for each $(t, \mathbf{y}) \in \mathcal{T} \times \Re^2$, $(X^{t,\mathbf{y}}(t), \mathbf{Z}^{t,\mathbf{y}}(t))$ *is the unique solution of the following BSDE:*

$$\begin{aligned} -dX^{t,\mathbf{y}}(s) &= -\widetilde{\lambda}(s, \mathbf{Y}(\cdot), \pi^*(s), \boldsymbol{\theta}^*(s))ds - (\mathbf{Z}^{t,\mathbf{y}}(s))' d\mathbf{W}(s) \ , \quad s \in [t, T] \ , \\ \mathbf{X}^{t,\mathbf{y}}(T) &= h(\mathbf{Y}(T)) \ , \end{aligned}$$

then, for each $s \in [t, T]$,

$$\begin{aligned} X^{t,\mathbf{y}}(s) &= u(s, \mathbf{Y}^{t,\mathbf{y}}(s)) \ , \\ \mathbf{Z}^{t,\mathbf{y}}(s) &= \partial_{\mathbf{y}} u(s, \mathbf{Y}^{t,\mathbf{y}}(s)) \mathbf{\Sigma}(s, \mathbf{Y}^{t,\mathbf{y}}(\cdot), \pi^*(s), \boldsymbol{\theta}^*(s)) \ . \end{aligned}$$

§5. Special Cases

- Quadratic penalty function:

$$\lambda(t, \mathbf{Y}(\cdot), \pi(t), \boldsymbol{\theta}(t)) := \frac{1}{2(1-\gamma(t))}||\boldsymbol{\theta}(t)||^2 Y_1(t) \ ,$$

  where $1-\gamma(t)$ is a measure of an insurer's relative risk aversion at time t and $\gamma(t) < 1$, $\forall t$.

- The first-order condition for maximizing $H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi(t), \boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ gives the following pair of equations:

$$\begin{aligned}
\partial_{\theta_1} H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi(t), \boldsymbol{\theta}) &= Y_1(t)\left(-\pi(t)\sigma(t) + \rho_{12}(t)\sigma_Y - \frac{\theta_1}{1-\gamma(t)}\right) = 0 \ , \\
\partial_{\theta_2} H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi(t), \boldsymbol{\theta}) &= Y_1(t)\left(\sigma_Y - \rho_{12}(t)\sigma(t)\pi(t) - \frac{\theta_2}{1-\gamma(t)}\right) = 0 \ .
\end{aligned}$$

- The first-order condition for minimizing $H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi, \boldsymbol{\theta}(t))$ with respect to π gives the following equation:

$$\partial_\pi H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi, \boldsymbol{\theta}(t)) = Y_1(t)(\mu(t) - r(t) + \theta_1(t)\sigma(t) + \rho_{12}(t)\theta_2(t)\sigma(t)) = 0 \ .$$

- The optimal portfolio strategy π^* of the insurer:

$$\pi^*(t) = \frac{\mu(t) - r(t) + \rho_{12}(t)\sigma(t)\sigma_Y(2 - \gamma(t))}{\sigma^2(t)(1 + \rho_{12}^2(t))(1 - \gamma(t))} \ .$$

- When $\rho_{12}(t) = 0$, $\forall t$, this becomes the "generalized" Merton ratio:

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)(1 - \gamma(t))} \ .$$

- The optimal strategy $\boldsymbol{\theta}^* := (\theta_1^*, \theta_2^*)$ of the market:

$$\theta_1^*(t) = \frac{r(t) - \mu(t) + \rho_{12}(t)\sigma(t)\sigma_Y[(1-\gamma(t))\rho_{12}^2(t) - 1]}{\sigma(t)(1+\rho_{12}^2(t))},$$

$$\theta_2^*(t) = \frac{\sigma_Y\sigma(t)(1-\gamma(t)-\rho_{12}^2(t)) - \rho_{12}(t)(\mu(t)-r(t))}{\sigma(t)(1+\rho_{12}^2(t))}.$$

The value function of the game satisfies the following BSDE:

$$\begin{aligned}
& -dX(t) \\
= \; & -Y_1(t)\Big(\kappa\mu_Y + r(t)Y_2(t) + \pi^*(t)(\mu(t)-r(t)) + \theta_1^*(t)\pi^*(t)\sigma(t) - \sigma_Y\theta_2^*(t) \\
& +\rho_{12}(t)\theta_2^*(t)\sigma(t)\pi^*(t) - \rho_{12}(t)\theta_1^*(t)\sigma_Y + \frac{1}{2(1-\gamma(t))}[(\theta_1^*(t))^2 + (\theta_2^*(t))^2]\Big)dt \\
& -\mathbf{Z}'(t)d\mathbf{W}(t), \quad X(T) = h(\mathbf{Y}(T)).
\end{aligned}$$

- Coherent risk measure:

$$\lambda(t, \mathbf{Y}(\cdot), \pi(t), \boldsymbol{\theta}(t)) = h(\mathbf{Y}(T)) = 0 \ .$$

- Suppose further that $\theta_1(t) = \theta_2(t) = \theta(t)$.

- The first-order condition for maximizing $H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi(t), \theta)$ with respect to θ gives:

$$\pi^*(t) = \frac{\sigma_Y}{\sigma(t)} \ .$$

- Intuition: Invest in the share to hedge the insurance risk

- The first-order condition for minimizing $H(t, \mathbf{Y}(\cdot), \mathbf{z}, \pi, \theta(t))$ with respect to π then gives:

$$\theta^*(t) = \frac{r(t) - \mu(t)}{(1 + \rho_{12}(t))\sigma(t)} \ .$$

=> Market price of risk for both financial and insurance risks

- The value function of the game satisfies the following BSDE:

$$\begin{aligned} -dX(t) &= -Y_1(t)\Big(\kappa\mu_Y + r(t)Y_2(t) + \pi^*(t)(\mu(t) - r(t)) + \theta^*(t)(\pi^*(t)\sigma(t) \\ &\quad -\sigma_Y + \rho_{12}(t)\sigma(t)\pi^*(t) - \rho_{12}(t)\sigma_Y)\Big)dt - \mathbf{Z}'(t)d\mathbf{W}(t) \ , \\ X(T) &= 0 \ . \end{aligned}$$

§6. Summary

- Adopted the BSDE approach to discuss a risk-based, optimal investment problem of an insurer

- Determined an optimal portfolio strategy to minimize the convex risk measure on terminal wealth

- Formulated the problem as a stochastic differential game and solved it using the BSDE approach

- Obtained closed-form and intuitively appealing solutions in some special cases

$\sim$ **Thank you !** $\sim$

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