A BSDE Approach to a Risk-Based Optimal Investment of an Insurer

Robert J. Elliott *  Tak Kuen Siu †

*Haskayne School of Business, University of Calgary, CANADA; School of Mathematical Sciences, University of Adelaide, AUSTRALIA
†Department of Applied Finance and Actuarial Studies, Faculty of Business and Economics, Macquarie University, Sydney, AUSTRALIA
§1. Optimal Investment of Insurers: Background and Literature

• Asset liability management for insurance businesses

• Traditional investment assets: Fixed interest securities

• Nowadays, investment items include risky assets like equities, share indices, derivatives, and many others.

• Quantitative Models: Make rational, scientific and justified investment decisions
• Optimal asset allocation in financial economics: Markowitz (1952), Samuelson (1969) and Merton (1971)

• Insurance risk attributed to liabilities or claims is not included.


• Stochastic Models for both insurance and financial risks are required.

• Insurance Risk Processes: Compound Poisson processes, diffusion approximations, jump-diffusion processes

• Objective functions: Minimize ruin probability, maximizing expected utility

• Key method: HJB dynamic programming approach
Risk measures in finance

- Value at Risk (VaR): popular, but not sub-additive and time-consistent, as well as leads to some bizarre and sub-optimal decisions if used as a binding constraint in portfolio selection, (Basak and Shapiro (2001))

- Coherent risk measures by Artzner et al. (1999): a remedy for some defects of VaR, but cannot incorporate liquidity risk of a large trading position

- Convex risk measures by Frittelli and Rosazza-Gianin (2002) and Föllmer Schied (2002): Generalization of coherent risk measures by incorporating liquidity risk, a key issue as highlighted by the recent global financial crisis
Keypoints of our work

• Discuss a risk-based, optimal investment problem for an insurer using a BSDE approach

• Use a convex risk measure as the objective function

• Formulate the problem as a two-player, zero-sum, stochastic differential game

• Solve the game problem using the BSDE approach

• Closed-form solutions to the optimal strategies
§2. The Model Dynamics

- Consider a continuous-time economy with two investment assets, namely, a bond $B$ and a share $S$

- The price process of $B$ evolves over time as:

$$B(t) = \exp \left( \int_0^t r(u) du \right), \quad B(0) = 1,$$

where $r(t)$ is the risk-free interest rate at time $t$

- The price process of $S$ is governed by a GBM:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW_1(t),$$

where $\{W_1(t)\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$. 
• For each $t \in [0,T]$, let $N(t)$ be the number of claims in the time interval $[0,t]$ and $Y_i$ be the size of the $i^{th}$ individual claim, where $i = 1, 2, \ldots$.

• We suppose that under $P$:

1. $\{N(t)| t \in \mathcal{T}\}$ is a Poisson process with a constant intensity $\lambda$;

2. $\{Y_i| i = 1, 2, \cdots\}$ are independent and identically distributed, (i.i.d.), nonnegative random variables with a common continuous distribution function $F(y)$ having the first and second moments $\mu_Y$ and $\sigma_Y^2$, respectively, where $\mu_Y, \sigma_Y < \infty$;

3. $\{N(t)| t \in \mathcal{T}\}$, $\{W_1(t)| t \in \mathcal{T}\}$ and $\{Y_i| i = 1, 2, \cdots\}$ are independent.
• The aggregate claims amount up to time $t$ is:

$$ Z(t) = \sum_{i=1}^{N(t)} Y_i . $$

• The premium rate associated with the relative security loading $\kappa$:

$$ c(\kappa) = (1 + \kappa) \lambda \mu Y , $$

where $\kappa > 0$.

• The classical Cramer-Lundberg model for the surplus process without investment:

$$ U^\kappa(t) = c(\kappa)t - Z(t) , \quad U^\kappa(0) = u_0 . $$
• Without loss of generality, we suppose $\lambda = 1$.

• **Theorem 1 (Grandell, 1991):** Let $\{W^2(t)\}$ be a second standard Brownian motion on $(\Omega, \mathcal{F}, P)$. Write $\mathcal{D}[0,T]$ for the space of càdlàg functions on $[0,T]$ endowed with the Skorohod topology. Define the diffusion process $\{R(t)\}$ by:

$$R(t) = u_0 + \mu_Y t + \sigma_Y W_2(t).$$

Then

$$\{\kappa U^\kappa(t/\kappa^2)\} \to \{R(t)\},$$

in $\mathcal{D}[0,T]$ as $\kappa \to 0$, where the convergence is in the sense of Skorohod topology.

• Assume that $\text{Cov}(W_1(t), W_2(t)) = \int_0^t \rho_{12}(u)du$. 
Information Structure

- \( \{ \mathcal{F}_R^R(t) \} \): The \( P \)-completed, right-continuous, natural filtration generated by the insurance risk process

- \( \{ \mathcal{F}_S^S(t) \} \): The \( P \)-completed, right-continuous, natural filtration generated by the share price process

- \( \mathcal{G}(t) = \mathcal{F}_R^R(t) \lor \mathcal{F}_S^S(t) \)

- \( \{ \mathcal{G}(t) \} \): The observable flow of information
• $\pi(t)$: The amount of money the insurer invests in the share at time $t$

• Let $\sigma(t, \pi(t)) := (\sigma(t)\pi(t), \sigma_Y)'$ and $W(t) := (W_1(t), W_2(t))'$. Then the evolution of the surplus process of the insurer associated with the investment process $\{\pi(t)\}$ over time is:

$$dV(t) = (\kappa \mu_Y + r(t)V(t) + \pi(t)(\mu(t) - r(t)))dt + \sigma'(t, \pi(t))dW(t),$$

$$V(0) = v_0.$$
A portfolio process $\pi$ is said to be admissible, (i.e. $\pi \in \mathcal{A}$), if it satisfies the following conditions:

1. $\pi$ is $\{\mathcal{G}(t)\}$-progressively measurable;

2. $$\int_0^T [\pi(t)]^2 dt < \infty, \quad \mathcal{P}\text{-a.s.};$$

3. the stochastic differential equation for the surplus process $V$ has a unique strong solution;

4. $$\int_0^T \left( |\kappa \mu_Y + r(t)V(t) + \pi(t)(\mu(t) - r(t))| + ||\sigma'(t, \pi(t))|| 
+ ||\sigma'(t, \pi(t))||^2 \right) dt < \infty, \quad P\text{-a.s.}$$
§3. Risk-based Optimal Investment for an Insurer

- **Definition 1** Let $S$ be the space of all lower-bounded random variables on the measurable space $(\Omega, \mathcal{G}(T))$. A convex risk measure $\rho$ is a functional $\rho : S \to \mathbb{R}$ which satisfies the following three properties:

1. If $X \in S$ and $K \in \mathbb{R}$, then
   \[ \rho(X + K) = \rho(L) - K. \]

2. For any $X_1, X_2 \in S$, if $X_1(\omega) \leq X_2(\omega)$, for all $\omega \in \Omega$, then
   \[ \rho(X_1) \geq \rho(X_2). \]

3. For any $X_1, X_2 \in S$ and $\zeta \in (0, 1)$,
   \[ \rho(\zeta X_1 + (1 - \zeta)X_2) \leq \zeta\rho(X_1) + (1 - \zeta)\rho(X_2). \]
• **Theorem 2** (Föllmer and Schied, 2002, and Frittelli and Rosazza-Gianin, 2002): Let $\mathcal{M}_a$ be a family of probability measures $Q$ which are absolutely continuous with respect to $P$. Define a function $\eta : \mathcal{M}_a \to \mathbb{R}$ such that $\eta(Q) < \infty$, $\forall Q \in \mathcal{M}_a$. Then for any convex risk measure $\rho(X)$ of $X \in S$, there exists a family $\mathcal{M}_a$ and a function $\eta$ such that

$$\rho(X) = \sup_{Q \in \mathcal{M}_a} \{ E_Q[-X] - \eta(Q) \}.$$

Here $E_Q$ represents expectation under $Q$.

• We must specify the family of probability scenarios $\mathcal{M}_a$ and the penalty function $\eta$. 
• Generate a family $\mathcal{M}_a$ by a set of probability measures of Girsanov transformation type

• Let $\{\theta(t)\}$ be a $\{G(t)\}$-predictable process, where $\theta(t) := (\theta_1(t), \theta_2(t)) \in \mathbb{R}^2$, such that

$$\int_0^T ||\theta(t)||^2 dt < \infty, \ P\text{-a.s.}$$

• Consider the $\{G(t)\}$-adapted process $\{\Lambda^\theta(t)\}$ defined by:

$$\Lambda^\theta(t) := 1 + \int_0^t \Lambda^\theta(u) \theta'(u) dW(u) .$$
• A process $\theta$ is an admissible scenario if

1. $\{\Lambda^\theta(t)\}$ is a $(G, P)$-martingale;

2. 

\[
\int_0^T ||\theta(t)||^2 dt < \infty , \quad \int_0^T ||\Lambda^\theta(t)\theta(t)|| dt < \infty , \quad P\text{-a.s.}
\]

• Write $\Theta$ for the space of admissible strategies $\theta$. 
• For each $\theta \in \Theta$, a new probability measure $Q_\theta \ll P$ on $\mathcal{G}(T)$ is defined by putting:

$$\frac{dQ_\theta}{dP} \bigg|_{\mathcal{G}(T)} := \Lambda^\theta(T) .$$

• Then we set

$$\mathcal{M}_a = \mathcal{M}_a(\Theta) := \{Q_\theta | \theta \in \Theta \} .$$
• Define, for each \((\pi, \theta) \in A \times \Theta\), a controlled process \(\{Y(t)\}\), where

\[
dY_1(t) = Y_1(t)\theta'(t)dW(t),
\]

\[
dY_2(t) = (\kappa \mu_Y + r(t)Y_2(t) + \pi(t)(\mu(t) - r(t)))dt + \sigma'(t, \pi(t))dW(t),
\]

so that \(Y(t) = (Y_1(t), Y_2(t)) = (\Lambda^\theta(t), V^\pi(t)) \in \mathbb{R}^2\).

• Consider the following penalty function \(\eta\):

\[
\eta(Q_\theta, \pi) := \mathbb{E}\left[\int_0^T \lambda(t, Y(\cdot), \pi(t), \theta(t))dt + h(Y(T))\right],
\]

\(\forall (\pi, \theta) \in A \times \Theta\),

for some bounded, measurable convex functions \(\lambda\) and \(h\).
• A convex risk measure can then be defined as:

\[ \rho(Y_2(T)) := \sup_{\theta \in \Theta} \{ \mathbb{E}^{\theta}[-Y_2(T)] - \eta(Q_\theta, \pi) \}, \]

where \( \mathbb{E}^{\theta} \) is expectation under \( Q_\theta \).

• The optimal investment problem of the insurer as a zero-sum stochastic differential game between the insurer and the market:

\[ \Phi(y_2) := \inf_{\pi \in \mathcal{A}} \inf_{\pi \in \mathcal{A}} \left\{ \sup_{\theta \in \Theta} \{ \mathbb{E}^{\theta}[-Y_2(T)] - \eta(Q_\theta, \pi) \} \right\}. \]
§4. The BSDE Solution to the Game Problem

- **Definition 2** Let $\xi(T)$ be a real-valued, square-integrable, $\mathcal{G}(T)$-measurable terminal condition and $g : \Omega \times T \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be a real-valued, $\{\mathcal{G}(t)\}$-progressively measurable, square-integrable, driver function. Then a solution of the BSDE associated with the driver function $g$ and the terminal condition $\xi(T)$ is a pair of square-integrable, $\mathcal{G}$-progressively measurable, processes $\{(X(t), Z(t))\}$ such that

$$X(t) = \xi(T) + \int_t^T g(u, X(u), Z(u))du - \int_t^T Z'(u) dW(u) .$$

Or equivalently, in a differential form,

$$dX(t) = -g(t, X(t), Z(t))dt + Z'(t)dW(t) , \quad X(T) = \xi(T) .$$
Theorem 3 (Existence and Uniqueness): Suppose the following conditions hold:

1. \( \{g(\omega,t,0,0)\} \) is a real-valued, square-integrable, \( \{G(t)\} \)-progressively measurable process;

2. there exists a positive constant \( K \) such that \( \forall (x_1, x_2, z_1, z_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2, \)

\[ |g(\omega,t,x_1,z_1) - g(\omega,t,x_2,z_2)| \leq K(|x_1 - x_2| + ||z_1 - z_2||), \]

\((l \otimes P)\text{-a.e.},\)

where \( l \) is the Lebesgue measure on \( T. \)

Then the BSDE associated with the driver function \( g \) and the terminal condition \( \xi(T) \) has a unique solution.
• **Theorem 4 (Comparison Theorem):** Suppose

1. \((X_1, Z_1)\) and \((X_2, Z_2)\) are the solutions of the two BSDEs associated with \((g_1, \xi_1(T))\) and \((g_2, \xi_2(T))\), respectively;

2. \(g_1\) and \(g_2\) satisfy the two conditions in Theorem 3.

If \(\xi_2(T) \leq \xi_1(T), P\text{-a.s., and }\forall (x, z) \in \mathbb{R} \times \mathbb{R}^2, g_2(\omega, t, x, z) \geq g_1(\omega, t, x, z), l \otimes P\text{-a.e., then }\forall (x, z) \in \mathbb{R} \times \mathbb{R}^2,

\[ X_2(\omega, t) \leq X_1(\omega, t), \forall t \in T, P\text{-a.s.} \]
• Define a double-indexed process \( \{\Gamma(t,u)|t \leq u\} \) as an adjoint process satisfying the following forward linear SDE:

\[
d\Gamma(t,u) = \Gamma(t,u)(\beta(u)du + \gamma'(u)dW(u)) , \quad \Gamma(t,t) = 1 ,
\]

so that \( \Gamma(t,u) \) satisfies the following semi-group property:

\[
\Gamma(t,s)\Gamma(s,u) = \Gamma(t,u) , \quad \forall t \leq s \leq u , P\text{-a.s.}
\]
• Theorem 5 (El Karoui et al., 1997) Let \( \{\beta(t)\} \) and \( \{\gamma(t)\} \) be bounded, \( \mathbb{R} \)-valued and \( \mathbb{R}^2 \)-valued, respectively, \( \{G(t)\} \)-progressively measurable processes. Suppose \( \{\alpha(t)\} \) is a real-valued, square-integrable, \( \{G(t)\} \)-progressively measurable process; \( \xi(T) \) is a real-valued, square-integrable, \( G(T) \)-measurable random variable. Consider the following linear BSDE:

\[
\begin{align*}
    dX(t) & = -(\alpha(t) + \beta(t)X(t) + \gamma'(t)Z(t))dt + Z'(t)dW(t), \\
    X(T) & = \xi(T).
\end{align*}
\]

Then this linear BSDE has a unique, square-integrable and \( \{G(t)\} \)-progressively measurable solution \( (X, Z) \), and \( X \) has the following expectation representation:

\[
X(t) = E\left[ \xi(T)\Gamma(t,T) + \int_t^T \Gamma(t,u)\alpha(u)du \bigg| G(t) \right].
\]
• **Lemma 1** Write

\[ \tilde{\lambda}(t, Y(\cdot), \pi(t), \theta(t)) := Y_1(t)(\kappa \mu_Y + r(t)Y_2(t) + \pi(t)(\mu(t) - r(t)) + \theta_1(t)\sigma(t)\pi(t) - \sigma_Y \theta_2(t) + \rho_{12}(t)\theta_2(t)\sigma(t)\pi(t) - \rho_{12}(t)\theta_1(t)\sigma_Y) + \lambda(t, Y(\cdot), \pi(t), \theta(t)) \],

and

\[ \tilde{J}^{\pi, \theta}(y) := E^y \left[ - \int_0^T \tilde{\lambda}(t, Y(\cdot), \pi(t), \theta(t)) dt - h(Y(T)) \right] , \]

where \( E^y \) is the conditional expectation given \( Y(0) = y \) under \( P \).

Then the stochastic differential game is equivalent to:

\[ \tilde{\Phi}(y) = \tilde{J}^{\pi^*, \theta^*}(y) = \inf_{\pi \in \mathcal{A}} \sup_{\theta \in \Theta} \tilde{J}^{\pi, \theta}(y) . \]
• Define the Hamiltonian \( H \) of the game problem by:

\[
H(t, Y(\cdot), z, \pi(t), \theta(t)) := -\tilde{\lambda}(t, Y(\cdot), \pi(t), \theta(t)).
\]

• **Definition 3 (Isaacs’ condition):** The Hamiltonian \( H \) is said to satisfy Isaacs’ condition if for a.s. all \( t \) and \( \omega \),

\[
\inf_{\pi \in A} \sup_{\theta \in \Theta} H(t, Y(\cdot), z, \pi(t), \theta(t)) = \sup_{\theta \in \Theta} \inf_{\pi \in A} H(t, Y(\cdot), z, \pi(t), \theta(t)).
\]

• **Lemma 2 ((Friedman, 1975, and Elliott, 1976):** There exist two measurable, \( \{G(t)\} \)-adapted, functions \( \pi^* \) and \( \theta^* \) such that

\[
H(t, Y(\cdot), z, \pi^*(t, Y(\cdot), z), \theta) \leq H(t, Y(\cdot), z, \pi^*(t, Y(\cdot), z), \theta^*(t, Y(\cdot), z)) \leq H(t, Y(\cdot), z, \pi, \theta^*(t, Y(\cdot), z)) , \quad \forall (\pi, \theta) \in A \times \Theta,
\]

if and only if \( H \) satisfies Isaacs’ condition.
• **Lemma 3** \( H(t, y(\cdot), z, \pi^*(t, y(\cdot), z), \theta^*(t, y(\cdot), z)) \) is Lipschitz in \( z \), uniformly in \((t, y(\cdot))\).

• **Theorem 6** Suppose Isaacs’ condition and the two conditions in Theorem 3 hold. Then there is a unique solution \( \{(X(t), Z(t))|t \in T\} \) of the BSDE associated with the driver function \( H(t, y(\cdot), z, \pi^*(t, y(\cdot), z), \theta^*(t, y(\cdot), z)) \) and terminal condition \( h(Y(T))\):

\[
-dX(t) = H(t, y(\cdot), z, \pi^*(t, y(\cdot), z), \theta^*(t, y(\cdot), z))dt - Z'(t)dW(t),
\]

\[
X(T) = h(Y(T)).
\]

Further the pair of strategies \((\pi^*(t, y(\cdot), z), \theta^*(t, y(\cdot), z))\) is a saddle point of the zero-sum stochastic differential game, and

\[
X(0) = \tilde{J}^{\pi^*, \theta^*}(y) = \inf_{\pi \in \mathcal{A}} \sup_{\theta \in \Omega} \tilde{J}^{\pi, \theta}(y) = \sup_{\theta \in \Omega} \inf_{\pi \in \mathcal{A}} \tilde{J}^{\pi, \theta}(y).
\]
• Suppose the control and the controlled state process are Markov.

• Write, for each \((t, y) \in \mathcal{T} \times \mathbb{R}^2\),
  \[
  \Sigma(t, y, \pi^*(t), \theta^*(t)) := (y_1(\theta^*(t))', \sigma'(\pi^*(t)))' \in \mathbb{R}^2 \otimes \mathbb{R}^2,
  \]
  \[
  A(t, y, \pi^*(t), \theta^*(t)) := \Sigma(t, y, \pi^*(t), \theta^*(t)) \Sigma'(t, y, \pi^*(t), \theta^*(t)) \in \mathbb{R}^2 \otimes \mathbb{R}^2.
  \]

• Define the following second-order partial differential operator:
  \[
  \mathcal{A}_{t, y}(\pi^*, \theta^*) := \sum_{i, j=1}^{n} a_{ij}(t, y, \pi^*(t), \theta^*(t)) \partial_{y_i y_j}^2 + (\kappa \mu_Y + r y_2 \pi^*(t)(\mu - r)) \partial_{y_2}.
  \]

• We now relate the BSDE solution of the stochastic differential game to a classical solution of a PDE using the nonlinear Feynman-Kac formula.
• **Theorem 7** Let $u(\cdot, \cdot) \in C^{1,2}(\mathcal{T} \times \mathbb{R}^2)$. Suppose there is a constant $K$ such that for each $(t, y) \in \mathcal{T} \times \mathbb{R}^2$,

$$|u(t, y)| + ||\partial_y u(t, y)\Sigma(t, y, \pi^*(t), \theta^*(t))|| \leq K(1 + ||y||) .$$

Assume further that $u(t, y)$ is the solution of the semi-linear parabolic PDE:

$$\partial_t u(t, y) + A_{t,y}(\pi^*, \theta^*)[u(t, y)] - \widetilde{\lambda}(t, y(\cdot), \pi^*(t), \theta^*(t)) = 0 ,$$

$$u(T, y) = h(y) .$$

If, for each $(t, y) \in \mathcal{T} \times \mathbb{R}^2$, $(X_{t,y}(t), Z_{t,y}(t))$ is the unique solution of the following BSDE:

$$-dX_{t,y}(s) = -\widetilde{\lambda}(s, Y(\cdot), \pi^*(s), \theta^*(s))ds - (Z_{t,y}(s))'dW(s) , \quad s \in [t, T] ,$$

$$X_{t,y}(T) = h(Y(T)) ,$$

then, for each $s \in [t, T]$,

$$X_{t,y}(s) = u(s, Y_{t,y}(s)) ,$$

$$Z_{t,y}(s) = \partial_y u(s, Y_{t,y}(s))\Sigma(s, Y_{t,y}(\cdot), \pi^*(s), \theta^*(s)) .$$
§5. Special Cases

• Quadratic penalty function:

\[ \lambda(t, Y(\cdot), \pi(t), \theta(t)) := \frac{1}{2(1 - \gamma(t))} \|\theta(t)\|^2 Y_1(t) , \]

where \(1 - \gamma(t)\) is a measure of an insurer’s relative risk aversion at time \(t\) and \(\gamma(t) < 1, \forall t\).

• The first-order condition for maximizing \(H(t, Y(\cdot), z, \pi(t), \theta)\) with respect to \(\theta\) gives the following pair of equations:

\[
\begin{align*}
\partial_{\theta_1} H(t, Y(\cdot), z, \pi(t), \theta) &= Y_1(t) \left( -\pi(t)\sigma(t) + \rho_{12}(t)\sigma_Y - \frac{\theta_1}{1 - \gamma(t)} \right) = 0 , \\
\partial_{\theta_2} H(t, Y(\cdot), z, \pi(t), \theta) &= Y_1(t) \left( \sigma_Y - \rho_{12}(t)\sigma(t)\pi(t) - \frac{\theta_2}{1 - \gamma(t)} \right) = 0 .
\end{align*}
\]
• The first-order condition for minimizing $H(t, Y(\cdot), z, \pi, \theta(t))$ with respect to $\pi$ gives the following equation:

$$\partial_\pi H(t, Y(\cdot), z, \pi, \theta(t)) = Y_1(t)(\mu(t) - r(t) + \theta_1(t)\sigma(t) + \rho_{12}(t)\theta_2(t)\sigma(t)) = 0.$$ 

• The optimal portfolio strategy $\pi^*$ of the insurer:

$$\pi^*(t) = \frac{\mu(t) - r(t) + \rho_{12}(t)\sigma(t)\sigma_Y(2 - \gamma(t))}{\sigma^2(t)(1 + \rho_{12}^2(t))(1 - \gamma(t))}.$$ 

• When $\rho_{12}(t) = 0$, $\forall t$, this becomes the “generalized” Merton ratio:

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)(1 - \gamma(t))}.$$
The optimal strategy $\theta^* := (\theta_1^*, \theta_2^*)$ of the market:

\[
\begin{align*}
\theta_1^*(t) &= \frac{r(t) - \mu(t) + \rho_{12}(t)\sigma(t)\sigma_Y[(1 - \gamma(t))\rho_{12}^2(t) - 1]}{\sigma(t)(1 + \rho_{12}^2(t))}, \\
\theta_2^*(t) &= \frac{\sigma_Y\sigma(t)(1 - \gamma(t) - \rho_{12}^2(t)) - \rho_{12}(t)(\mu(t) - r(t))}{\sigma(t)(1 + \rho_{12}^2(t))}.
\end{align*}
\]

The value function of the game satisfies the following BSDE:

\[
\begin{align*}
-dX(t) &= -Y_1(t) \left( \kappa \mu_Y + r(t)Y_2(t) + \pi^*(t)(\mu(t) - r(t)) + \theta_1^*(t)\pi^*(t)\sigma(t) - \sigma_Y\theta_2^*(t) \\
&\quad + \rho_{12}(t)\theta_2^*(t)\sigma(t)\pi^*(t) - \rho_{12}(t)\theta_1^*(t)\sigma_Y + \frac{1}{2(1 - \gamma(t))}((\theta_1^*)^2 + (\theta_2^*)^2) \right) dt \\
&\quad - Z'(t)dW(t), \quad X(T) = h(Y(T)).
\end{align*}
\]
• Coherent risk measure:

\[ \lambda(t, Y(\cdot), \pi(t), \theta(t)) = h(Y(T)) = 0. \]

• Suppose further that \( \theta_1(t) = \theta_2(t) = \theta(t) \).

• The first-order condition for maximizing \( H(t, Y(\cdot), z, \pi(t), \theta) \)
  with respect to \( \theta \) gives:

\[ \pi^*(t) = \frac{\sigma_Y}{\sigma(t)}. \]

• Intuition: Invest in the share to hedge the insurance risk
• The first-order condition for minimizing $H(t, Y(\cdot), z, \pi, \theta(t))$ with respect to $\pi$ then gives:

$$\theta^*(t) = \frac{r(t) - \mu(t)}{(1 + \rho_{12}(t))\sigma(t)}.$$ 

$\Rightarrow$ Market price of risk for both financial and insurance risks

• The value function of the game satisfies the following BSDE:

\[-dX(t) = -Y_1(t)\left(\kappa\mu_Y + r(t)Y_2(t) + \pi^*(t)(\mu(t) - r(t)) + \theta^*(t)(\pi^*(t)\sigma(t) - \sigma_Y + \rho_{12}(t)\sigma(t)\pi^*(t) - \rho_{12}(t)\sigma_Y)\right)dt - Z'(t)dW(t),\]

$X(T) = 0.$
§6. Summary

- Adopted the BSDE approach to discuss a risk-based, optimal investment problem of an insurer

- Determined an optimal portfolio strategy to minimize the convex risk measure on terminal wealth

- Formulated the problem as a stochastic differential game and solved it using the BSDE approach

- Obtained closed-form and intuitively appealing solutions in some special cases
~ Thank you !~
References


