Will Alternative Investments Improve Your Portfolio?

by Craig F Ansley

Investors are increasingly investigating alternative investments such as private equity, hedge funds, infrastructure and commodities to increase returns and diversify risk. However, with conventional asset allocation techniques it can be difficult to decide whether a new asset class will improve the portfolio. These techniques are very sensitive to the return assumptions, which for alternative investments are often very uncertain.

Framing the Question

In this paper we show how to approach the problem by addressing the question: *What risk and return characteristics must a new asset class have to improve the current portfolio?*

We describe a simple graphical method that handles the uncertainty and gives a clear understanding of why a new asset class will or will not improve portfolio performance.

Conventional techniques are based around optimization programs that find the asset mix giving the highest expected return for a given level of risk. The programs depend on assumptions about return and volatility parameters for the new asset class, and about the pairwise correlations with all the asset classes currently in the portfolio. However, the results are notoriously sensitive to small changes in the input assumptions. This inherent weakness led Michaud (1989) to describe naively-used optimisers as "estimation-error maximizers".

For an alternative investment, the rage of possible input parameters is typically large, leading to wide swings in the outcomes that substantially diminish the usefulness of the conventional approach.

The method described below allows prudent decisions about alternative investments to be made without the precise information required by optimizers, by putting the uncertainty surrounding return parameters into a manageable context.

The Sharpe Ratio

We monitor portfolio improvement by changes in the *Sharpe ratio*:

$$S_p = \frac{r_p - r_f}{\sigma_p} \qquad (1)$$

where

$r_p = E(R_p)$ and $R_p =$ annual return on portfolio P

$r_f =$ risk free rate of interest

$\sigma_p =$ standard deviation (volatility) of R_p

The Sharpe ratio is a widely used measure of risk adjusted performance. If a portfolio change increases the Sharpe ratio, then the portfolio is improved in the sense that it can either be combined or levered with the risk-free asset to produce a new portfolio with the same risk but higher expected return than the original, or alternatively with the same return but lower risk. In practice there is no risk free asset, and cash usually serves as a proxy. In any case, if the initial change alters portfolio risk by only a small amount, the same result can often be obtained with an adjustment of the income/growth ratio. It is a simple matter to rearrange the new portfolio to give a better risk/return trade-off.

Will a new asset class improve the Sharpe ratio?

Suppose we currently hold portfolio *P*, and we are considering making a small allocation *w* to a new asset class *A*. Let r and σ be the expected annual return and volatility for *A* and let ρ be its return correlation with *P*. The correlation ρ between the returns on *A* and the portfolio *P* must be calculated from the correlations between the returns on *A* and each of the asset classes in *P*, and their volatilities.

If all existing allocations are reduced proportionately (this assumption is justified later), then we can in principle compute a new Sharpe ratio $S_p^A(r,\sigma,\rho,w)$, which depends on the return parameters r, σ and ρ and the weight *w*.

If the contemplated weight *w* is small, as it usually is for a new investment into and alternative asset class, then the Sharpe ratio will be improved if $S_p^A(r,\sigma,\rho,w)$ is an increasing function of *w* at $w=0$. In mathematical terms, the partial derivative $\partial S_p^A / \partial w > 0$ at $w=0$.

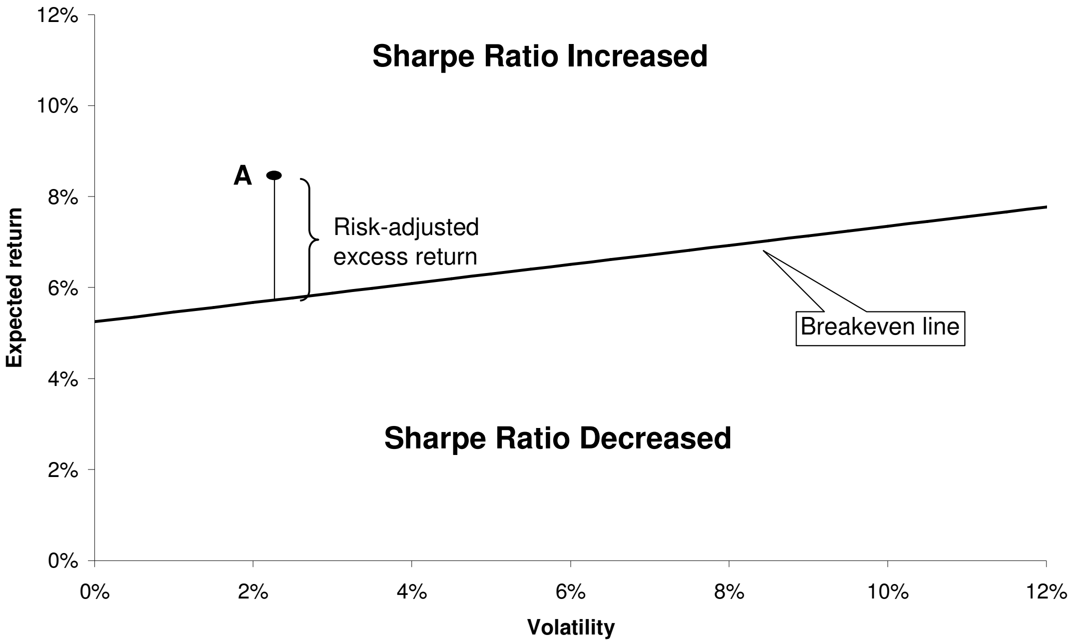
Of course, the new Sharpe ratio and therefore the partial derivative $\partial S_p^A / \partial w$ depend on the return parameters r, σ and ρ, and their values are uncertain. But if we assign a tentative value to one of the three parameters, usually ρ, with some algebraic manipulation we can find the breakeven relationship between the other two, usually r and σ, determined by setting $\partial S_p^A / \partial w = 0$. This breakeven line defines areas in which pairs of portfolio parameters, say r and σ, will improve the Sharpe ratio, as shown in Figure 1 below. Possible values of r and σ for *A* can then be examined to decide whether such values will improve the portfolio.

Example 1

In Figure 1, we have plotted the breakeven line for the addition of a new asset class *A* with mean return 8.5% p.a. and volatility 3.0% to a portfolio *P* as described in Appendix 1. The breakeven line was calculated assuming a correlation $\rho = 0.5$ between the returns for *A* and the portfolio *P*, and the risk-free rate is 5.25%. Because *A* lies above the breakeven line, a small allocation to *A* will improve the Sharpe ratio for the portfolio. In fact, the vertical distance of *A* above the line (in this case 2.6%) can be interpreted as the risk-adjusted excess return for *A*. Adding a small allocation to *A* is equivalent to adding a small allocation to an asset with zero volatility and return 2.6% above the risk free rate.

We note that Figure 1 does not allow for uncertainty in the return parameters; this is discussed below.

Figure 1
Example of Breakeven Criterion



The breakeven line

In Appendix 2, we show that the breakeven line is given by

$$r = r_p - S_p(\sigma_p - \rho\sigma) \qquad (2)$$

On the breakeven line the partial derivative $\partial S_p / \partial w$ of the Sharpe ratio S_p with respect to the portfolio weight *w* of the new asset class is zero at $w = 0$. An expression for $\partial S_p / \partial w$ is given in Appendix 2. For values of *r* and σ above the line, the partial derivative is positive, so that small allocations to *A* will improve the Sharpe ratio, while for values below the line, the partial derivative is negative and allocations to *A* will decrease the Sharpe ratio.

Some immediate implications of (2) are:

- For a given value of ρ, the breakeven line relating r to σ is a straight line with intercept $r_p - S_p\sigma_p = r_f$ (after substituting from (1)) and slope ρS_p.
- If the new asset class *A* has zero volatility (i.e. it is risk free) then it will improve the portfolio only if its return is greater than r_f. Otherwise, it would be better to invest in the risk free asset.
- If ρ is positive, the greater the volatility σ, the greater the excess return required for *A*.
- Conversely, if ρ is negative, the return hurdle becomes lower as σ increases, because it represents a stronger diversification benefit.
- The greater the correlation ρ between the returns on *A* and *P*, the greater the excess return hurdle for any given value of σ.

The last three implications deserve some more discussion. From Eq. (A2) in Appendix 2, for a small allocation w to the new asset, the return variance $\sigma^2(w)$ is given by

$$\sigma^2(w) \quad \approx \sigma_p{}^2\{1-2w(1-\rho\frac{\sigma}{\sigma_p})\} \ = \sigma_p{}^2(1-2w)+2w\rho\sigma\sigma_p \qquad (3)$$

When ρ is positive, the portfolio return variance $\sigma^2(w)$ and therefore the volatility $\sigma(w)$ of the portfolio increases as the volatility σ of the new asset class increases. If the portfolio volatility increases, we need a higher expected return to break even: the expected return contribution from the new asset class must be higher.

On the other hand, when ρ is negative, the portfolio volatility $\sigma(w)$ decreases as the volatility σ of the new asset class increases. There is a strong diversification effect. If the portfolio volatility decreases, we need a lower expected return to break even, and so the expected return contribution from the new asset class can be lower.

Finally, fixing the new asset class volatility σ in (3), the portfolio volatility $\sigma(w)$ increases as ρ increases, so that a higher expected return contribution is required to break even.

Another interpretation of the breakeven line

The breakeven line also has another interpretation. If we substitute the formula for S_p from (1) into (2), we obtain, with a little algebraic manipulation

$$r-r_f \quad = \quad \rho\frac{\sigma}{\sigma_p}(r_p-r_f) \quad = \quad \beta(r_p-r_f) \qquad (4)$$

where $\beta=\rho\sigma/\sigma_p$. Denoting the actual return on the new asset by R (as distinct from its expected return r), we recognize β as the coefficient of regression of the new asset's excess return $R-r_f$ on the portfolio excess return R_p-r_f.

If the portfolio was the market portfolio, (4) would represent the *security market line* relating the expected return on any security to the expected return on the market. No security could lie above the security market line, otherwise the market portfolio would not be efficient, and a portfolio with market risk but with expected return greater than market could be constructed. See Alexander *et al*, 2000, Chptr 10 for an exposition of this argument.

Exactly the same logic applies here. If the expected return on the new asset lies above the line (3), defined in terms of our portfolio P rather than the market, then a new portfolio could be constructed with the same risk as P but with higher expected return. We could therefore have obtained the breakeven line (2) by direct analogy with the arguments leading to the derivation of the security market line.

How should the new asset class be accommodated?

If the analysis shows that the new asset class will improve the Sharpe ratio, which of the existing asset classes should be down-weighted to accommodate it?

We show in Appendix 2 that the improvement in the Sharpe ratio is the same however the accommodation is made, provided that the existing portfolio is efficient.

From a practical standpoint, however, if we make a change that improves the Sharpe ratio, then we may have to rearrange the portfolio to get the (improved) risk/return trade-off we want. If we accommodate a new asset class with higher volatility, such as private equity, by replacing growth assets, or a lower volatility class like a fund of hedge funds by replacing income assets, the overall portfolio volatility will stay close to where it was.

No Risk Free Asset

The preceding arguments assume there is a risk-free asset. In this case, all efficient portfolios are a combination of the market portfolio and the risk-free asset. The problem is then to decide whether a new asset will change the market portfolio, or at what price the new asset will enter the market.

In practice, there is no risk-free asset with the properties assumed for the capital asset pricing model. However, our approach can still be applied. Mean-variance efficiency is obtained by maximising quadratic utility:

$$V = r - \lambda\sigma^2 \tag{5}$$

where r is the mean return, σ the volatility and λ the (constant) coefficient of risk aversion. The optimal portfolio is obtained by maximising V along the efficient frontier for the asset classes under consideration, so that

$$\frac{\partial V}{\partial \sigma} = \frac{\partial r}{\partial \sigma} - 2\lambda\sigma = 0 \tag{6}$$

and

$$\lambda = \frac{1}{2\sigma}\frac{\partial r}{\partial \sigma} \tag{7}$$

where $V(w) - V_p =$ and σ are constrained to the efficient frontier.

Suppose now that we form a new portfolio by allocating a small weight w to an alternative asset class and $(1-w)$ to the original portfolio. Let r and σ be the mean and volatility respectively of the alternative asset class and ρ be the correlation between the alternative class returns and the portfolio returns. Denote the existing portfolio parameters by the subscript p. As in Appendix 2, the expected return $r(w)$ and return variance $\sigma^2(w)$ for the new portfolio are given by:

$$r(w) = wr + (1-w)r_p \tag{8}$$

$$\sigma^2(w) \approx {\sigma_p}^2\{1 - 2w(1 - \rho\frac{\sigma}{\sigma_p})\} \tag{9}$$

If $V(w)$ is the utility for the new portfolio, then from (5), (8) and (9)

$$V(w) - V_p = w(r - r_p) + 2w\lambda\sigma_p^2\left(1 - \rho\frac{\sigma}{\sigma_p}\right) \tag{10}$$

The breakeven line for portfolio improvement is where $V(w) - V_p = 0$, so that from (10)

$$r = r_p - 2\lambda\sigma_p^2\left(1 - \rho\frac{\sigma}{\sigma_p}\right) \tag{11}$$

If the portfolio *P* is efficient then, using (6), (11) becomes

$$r = r_p - \sigma_p\frac{\partial r_p}{\partial\sigma_p}\left(1 - \rho\frac{\sigma}{\sigma_p}\right) = r_z + \rho\sigma\frac{\partial r_p}{\partial\sigma_p} \tag{12}$$

where $r_z = r_p - \sigma_p\frac{\partial r_p}{\partial\sigma_p}$ is the point at which the tangent line to the efficient frontier at *P* intersects the vertical axis $\sigma = 0$. The breakeven line is constructed as before, but with intercept r_z and slope $\rho\frac{\partial r_p}{\partial\sigma_p}$.

The argument in Appendix 2 can be easily modified to show that the improvement in the utility is the same however accommodation is made for the new asset, provided the existing portfolio is efficient.

Tangent slope

Because the portfolio *P* is on the efficient frontier then from (6)

$$\frac{\partial r_p}{\partial\sigma_p} = 2\lambda\sigma_p \tag{13}$$

This provides a very easy to compute the breakeven intercept r_z and slope $\rho\frac{\partial r_p}{\partial\sigma_p}$.

Example 2

This extends Example 1 to the case where there is no risk-free asset. Figure 2 shows the efficient frontier for the asset classes from which portfolio P is formed; see Appendix 1 for the assumptions. The tangent line $r = r_z + \sigma\frac{\partial r_p}{\partial\sigma_p}$ intersects the vertical axis at $r_z = 5.8\%$, considerably higher than the risk-free rate of 5.25% assumed in Example 1.

Figure 3 shows the breakeven line calculated from (12) above. The risk-adjusted excess return for asset *A* (see Example 1) is reduced to 2.2%.

Figure 2
Efficient Frontier and Tangent at Portfolio *P*

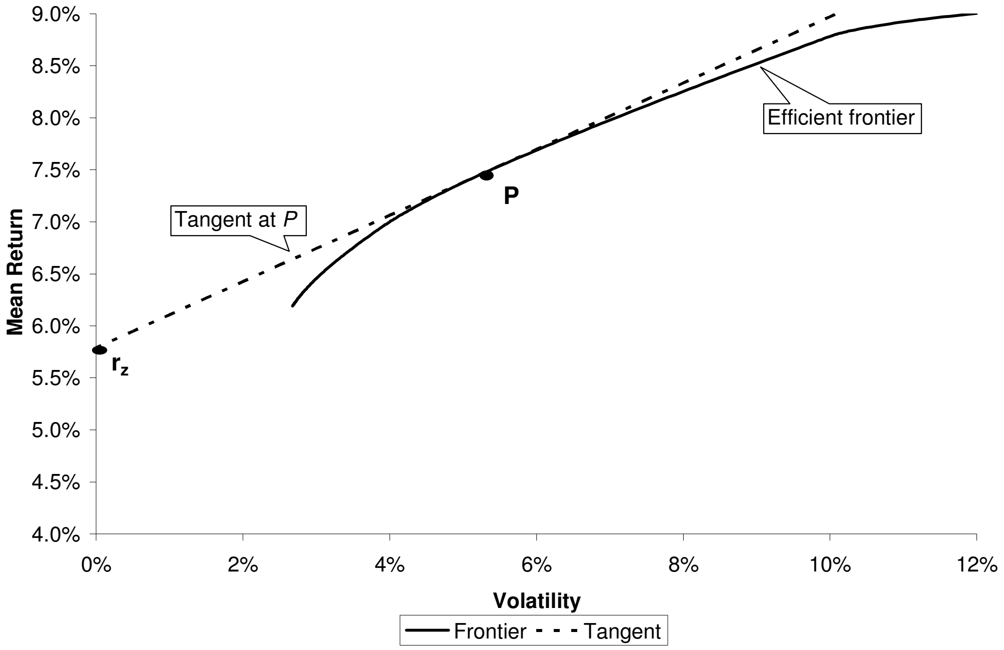
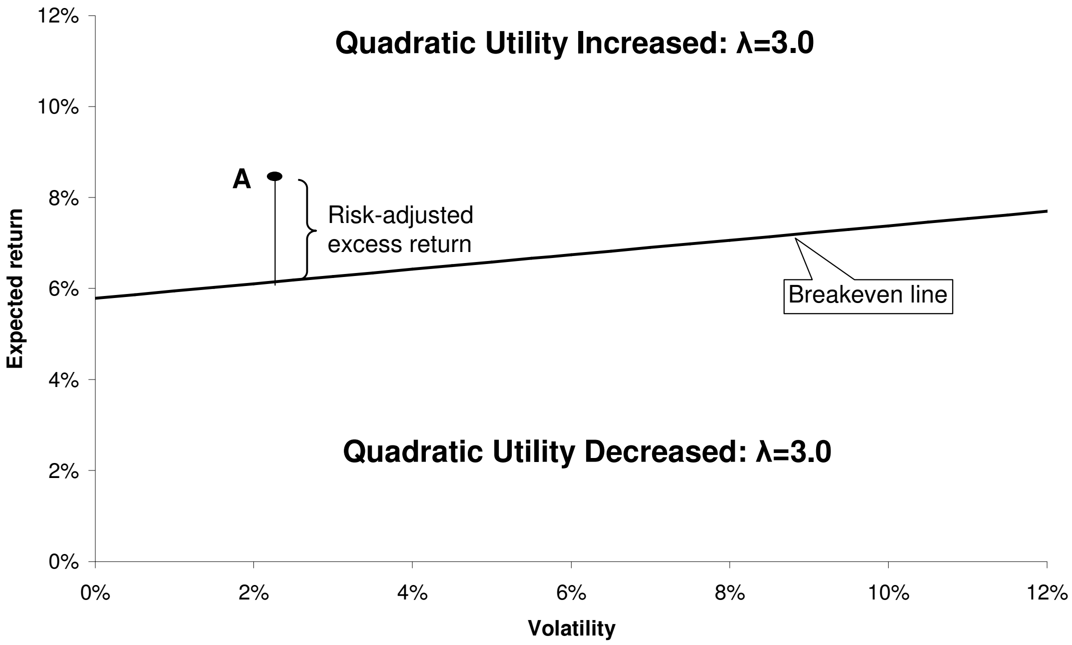


Figure 3
Breakeven Criterion for Example 2 with No Risk-Free Asset



Relationship with Sharpe ratio

The analysis in terms of the Sharpe ratio leading to (2) when there is a risk-free asset follows immediately from (12) by noting that in this case

$$\frac{\partial r_p}{\partial \sigma_p} = \frac{r_p - r_f}{\sigma_p}$$

whence

$$r = r_p - \sigma_p \frac{r_p - r_f}{\sigma_p}\left(1 - \rho\frac{\sigma}{\sigma_p}\right) = r_p - S_p(\sigma_p - \rho\sigma)$$

Uncertain return parameters

If we knew the return parameters r and σ for the new asset class *A*, and its correlation ρ with the portfolio *P*, it would be a simple matter to decide whether making an allocation to *A* would improve the portfolio. We would just plot the point (r, σ) as on Figure 1 and see if it was above the breakeven line.

Unfortunately, the return parameters for new asset classes are more often than not subject to considerable uncertainty. However, we should be able to define a region within which we can place the parameters with reasonable confidence. In fact, if this is not possible, there would appear to be no basis on which to consider the asset class in the first place.

Uncertain mean return and volatility

A feasible region for the mean return r and volatility σ might be set by using standard statistical methods to find the most likely values, plus optimistic and pessimistic estimates, and extending these to a feasible region, perhaps after modification for changed expectations for the future. This is the method used in the example below. More formal methods such as Bayesian or Stein estimation could also be applied; see e.g. Michaud (1998, Chptr 8).

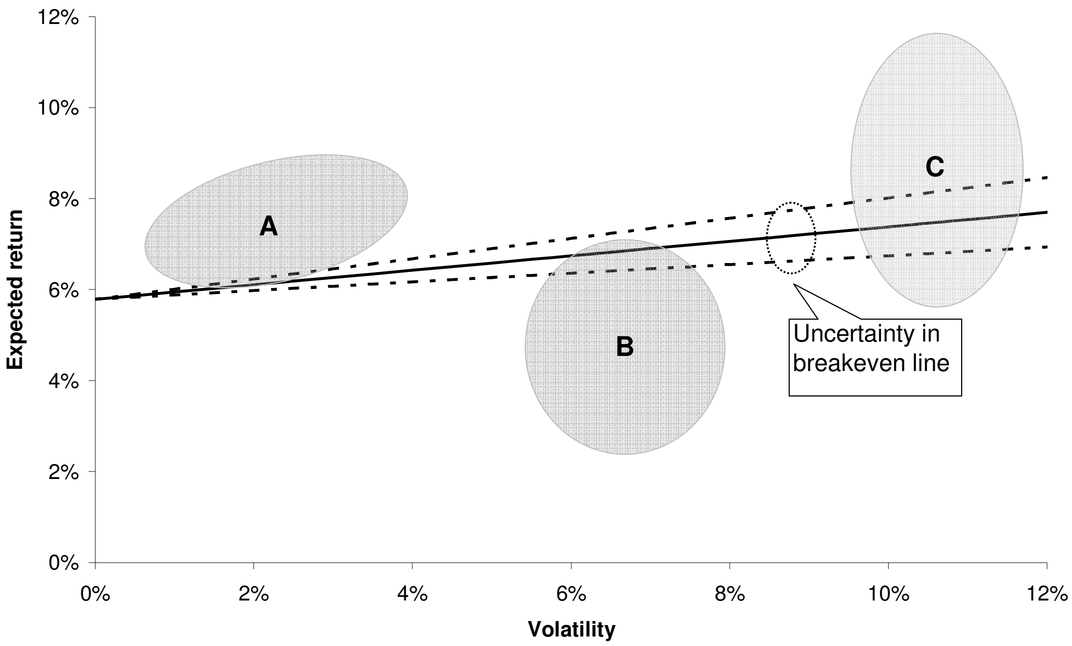
Examples of feasible regions for three candidate asset classes *A*, *B* and *C* are shown in Figure 4 below. These are discussed further below.

Uncertain correlations

The slope $\rho\frac{\partial r_p}{\partial \sigma_p}$ of the breakeven line depends on the correlation ρ between the new asset class and the current portfolio. Of course, ρ can be as difficult to estimate as the mean r and volatility σ. In practice, the breakeven line must be replaced by a band representing feasible values. Again, this can often be developed by looking at best estimates of the correlation together with optimistic and pessimistic estimates. An optimistic estimate is a smaller value, because it represents a better diversification opportunity. In terms of Figures 3 and 4, a smaller correlation would give a breakeven line with a smaller slope, and therefore a larger region for improving the utility.

A typical band for the breakeven line obtained from optimistic and pessimistic correlation estimates is shown in Figure 4.

Figure 4
Examples of Feasible Parameter Regions



Decision under uncertainty

In Figure 4, there are feasible regions for three asset classes *A*, *B* and *C*. There is also a band representing uncertainty in the breakeven line. For the sake of the example, we are assuming the same band applies to all three asset classes *A*, *B* and *C*. In practice, individual asset classes would be examined separately, and a separate band developed for each.

From Figure 4 we can infer the effects of small allocations to the candidate asset classes. *Asset class A* will very almost certainly improve the utility, because only the lower limits of the feasible expected return range lie below the breakeven band. *Asset class B* will almost surely decrease the utility, because most of the feasible region lies below the breakeven band.

Asset class C presents more of a problem. The feasible region is almost bisected by the upper boundary of the breakeven band, obtained by taking a pessimistic view of the correlation between the returns of *C* and the portfolio *P*. At the centre of the band improvement is still doubtful. We would be confident of increasing the Sharpe ratio only if we held an optimistic view of the correlation (lower limit of breakeven band). To proceed with an investment in *C*, more research would be needed to assure us that the return parameters were more likely to lie in the upper part of the feasible region, or that the volatility in future would be lower than our initial expectations.

Alternatively, we may be able to lower the volatility through a different implementation, such as a multi-manager fund. Provided the expected return range stays the same, reducing the volatility would shift the feasible region to the left, so that more of it lay above the breakeven band.

Example 3: CCFs

Suppose we are considering adding collateralized commodity futures (CCFs) to a balanced portfolio *P* with 30% US equities, 30% international equities (unhedged), and 40% bonds. These asset classes are benchmarked to the S&P 500, MSCI EAFE and the Lehman Brothers US Aggregate respectively. The return assumptions underlying this choice are shown in Table 1 below.

Table 1
Asset Class Assumptions for Example 3

Asset Class	Mean return (%)	Volatility (%)	Correlations		
			US eq	Int eq	Bonds
US equities	11.0	16.0	1.0	0.6	0.3
International equities	11.0	16.0	0.6	1.0	0.2
US bonds	7.0	6.0	0.3	0.2	1.0
			Portfolio P weights		
Portfolio P	*9.4*	*9.5*	*30%*	*30%*	*40%*

The assumptions in Table 1 are based on historical return statistics for the period January 1970 – March 2004. The benchmark portfolio *P* is approximately the optimal choice for risk aversion $\lambda = 2.0$. The efficient frontier and tangent line at *P* are sown in Figure 5 below.

Figure 5
Efficient Frontier for Example 3

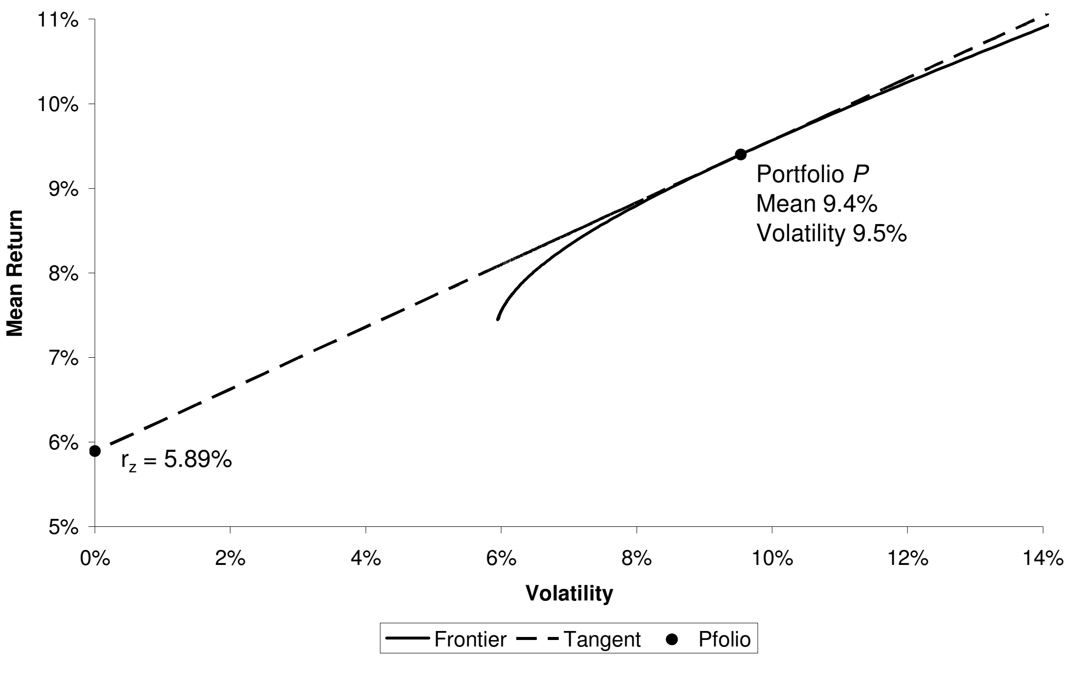


Table 2 summarises performance statistics over the period January 1970 – March 2004 for the Goldman Sachs Commodity Index (GSCI), the Dow Jones AIG Commodity Index (DJ AIG) and the Gorton-Rouwenhorst (2005) equally weighted commodity index (GREWI). The GSCI has no weighting constraints, and exhibits the highest mean return and volatility. The DJ AIG limits the weight in any one sector to 33%, and the GREWI is equally weighted across a wide range of futures contracts. The DJ AIG and GREWI have similar volatilities. The three indices have back histories of different lengths, but are highly correlated for overlapping data periods.

Before constructing feasible regions for CCF parameters from the data in Table 2, we note the following.

- The equity risk premium is unlikely to be as high in future as it was in the past. See Cornell (1999).
- It is evident that the choice of index is very important when implementing an allocation to CCFs. The GSCI with unconstrained weights is more volatile than the other two.
- Because there is no statistical method that can forecast risk premia and volatilities accurately for an asset class such as CCFs under changing market conditions, it is prudent to take a conservative approach.

Keeping these points in mind, reasonable feasible parameter regions can be constructed as in Table 3.

The break-even chart obtained by combining the two sets of parameter ranges in Table 3 is shown in Figure 5. The most likely means and volatilities from Table 3 are also plotted.

Table 2
Collateralized Commodity Futures Performance Statistics[1]

	Goldman Sachs Commodity Index (GSCI)	**Dow Jones AIG Commodity Index (DJ AIG)**	**Gorton-Rouwenhorst[1] Equally Weighted Index (GREWI)**
Back history to 2005 from:	1970	1991	1970
Nominal returns % pa			
Mean	13.1	8.3	12.1
95% conf interval	*(11.3, 14.9)*	*(6.5, 10.2)*	*(10.8, 13.3)*
Volatility	18.4	11.8	12.9
95% conf interval	*(17.1, 19.6)*	*(10.4, 13.0)*	*(12.0,13.8)*
Premium over bonds % pa			
Mean pa	4.4	0.7	3.4
95% conf interval	*(2.4, 6.3)*	*(-1.2, 2.6)*	*(1.9, 4.9)*
Correlation with benchmark portfolio	-0.03	0.18	0.06
95% conf interval	*(-0.12, 0.07)*	*(0.02, 0.33)*	*(-0.04, 0.15)*

[1]See Gorton & Rouwenhorst (2005)

Table 3
Feasible Parameter Ranges for CCFs

	Most likely	Optimistic	Pessimistic
GSCI:			
Premium over bonds	3%	5%	1%
Mean return	10%	13%	7%
Volatility	19%	17%	21%
Correl with benchmark	0.0	-0.1	0.1
DJ AIG:			
Premium over bonds	0%	2%	-2%
Mean return	7%	9%	5%
Volatility	12%	11%	13%
Correl with benchmark	0.2	0.0	0.4
GREWI:			
Premium over bonds	2%	4%	0%
Mean return	9%	11%	7%
Volatility	13%	12%	14%
Correl with benchmark	0.1	0.0	0.2

Figure 6
Quadratic Utility Breakeven Chart for CCFs

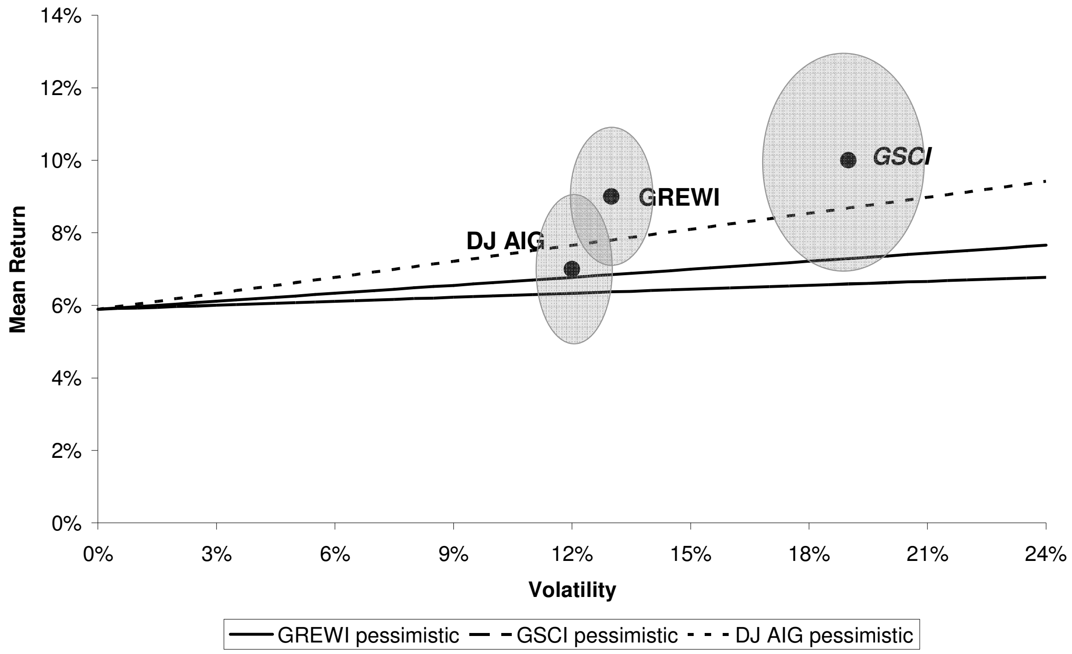


Figure 6 shows that making a small allocation to CCFs benchmarked to the GSCI is likely to improve quadratic utility, but there is unlikely to be an improvement with CCFs benchmarked to the DJ AIG. There is likely to be an improvement with CCFs benchmarked to the GREWI also, but this index is not currently maintained on a commercial basis.

For the DJ AIG, the most likely risk-adjusted excess return is -6%, with optimistic and pessimistic limits 3% and -15% respectively. For the GSCI, the most likely excess return is 4%, with optimistic and pessimistic limits 11% and -3% respectively.

Conclusion

The advantage of the method proposed here is that the effect of the parameter uncertainty becomes clear, allowing a decision to be made with this uncertainty taken properly into account. This approach reduces investor scepticism about introducing alternative investments by showing the degree of uncertainty and its effect under different possible scenarios. It gives a clear understanding of what risk and return characteristics must be exhibited by a new asset class to improve the current portfolio.

References

Alexander, G.J., Sharpe, W.F. & Bailey, J.V. (2000) *Fundamentals of Investments*, 3rd ed. Englewood Cliffs, NJ: Prentice Hall.

Cornell, B. (1999) *The Equity Risk Premium: The Long-Run Future of the Stock Market*. New York: Wiley.

Gorton G. & Rouwenhorst K.G. (2005) Facts and fantasies about commodities futures. Working paper.

Michaud, R.O. (1989) The Markowitz optimization enigma: Is 'optimised' optimal? *Financial Analysts Journal* **45** No.1, January-February, 31-42.

Michaud, R.O. (1998) *Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation*. Boston: Harvard Business School Press.

Sharpe, W.F. (1994) The Sharpe ratio. *Journal of Portfolio Management* **21** No.1 (Fall) 49-58.

Appendix 1—Asset class assumptions

The portfolio *P* used in the Examples 1 and 2 is formed from the asset classes whose return means, volatilities and correlations are given in Table A1 below. The benchmark asset allocation for portfolio P, together with the resulting mean return and volatility. All returns are assumed to be serially uncorrelated.

Portfolio *P* lies on the efficient frontier and is the optimal quadratic utility portfolio for $\lambda = 3.0$.

Table A1
Asset Class Assumption for Examples 1 and 2s

Asset Class	**Mean return %**	**Volatility %**	**Correlations**				
			DE	**GEH**	**DFI**	**GFHI**	**DC**
Domestic equities	9.0	17.0	1.0	0.4	-0.1	-0.2	-0.1
Global eq. hedged	8.5	14.9	0.4	1.0	0.0	0.1	0.2
Domestic fixed int	6.1	3.3	-0.1	0.0	1.0	0.4	0.3
Global fixed int hedged	6.1	3.3	-0.2	0.1	0.4	1.0	0.4
Domestic cash	5.5	1.5	-0.1	0.2	0.3	0.4	1.0
			Portfolio P asset class weights (%)				
Portfolio P	*7.5*	*5.3*	*30%*	*20%*	*27%*	*23%*	*0%*

Appendix 2—The breakeven line

Suppose that we form a new portfolio by allocating a small weight w to an alternative asset class and $(1-w)$ to the original portfolio. Let r and σ be the mean and volatility respectively of the alternative asset class and ρ be the correlation between the alternative class returns and the portfolio returns. Denote the portfolio parameters by the subscript p as in the text. The expected return $r(w)$ and return variance $\sigma^2(w)$ for the new portfolio are given by:

$$r(w) = wr + (1-w)r_p \tag{A1}$$

$$\begin{aligned}\sigma^2(w) &= w^2\sigma^2 + 2w(1-w)\rho\sigma\sigma_p + (1-w)^2\sigma_p^{\,2} \\ &= \sigma_p^{\,2} - 2w(\sigma_p^{\,2} - \rho\sigma\sigma_p) + w^2(\sigma^2 - 2\rho\sigma\sigma_p + \sigma_p^{\,2})\end{aligned}$$

We can discard the third term because w^2 is close to zero. Thus

$$\sigma^2(w) \approx \sigma_p^{\,2}\{1 - 2w(1 - \rho\frac{\sigma}{\sigma_p})\} \tag{A2}$$

The approximate equality here and below means that powers of w of two or higher have been omitted.

From the binomial theorem approximation $(1+x)^{1/2} \approx 1 - \frac{1}{2}x$ for small x we have

$$\frac{1}{\sigma(w)} \approx \frac{1}{\sigma_p}\{1 + w(1 - \rho\frac{\sigma}{\sigma_p})\} \tag{A3}$$

From (A1) and (A3) the Sharpe ratio $S_p^A(w)$ for the new portfolio is

$$\begin{aligned} S_p^A(w) &= \frac{r(w) - r_f}{\sigma(w)} \\ &\approx \frac{w(r - r_p) + r_p - r_f}{\sigma_p}\{1 + w(1 - \rho\frac{\sigma}{\sigma_p})\} \\ &\approx S_p + w\left\{S_p(1 - \rho\frac{\sigma}{\sigma_p}) + \frac{r - r_p}{\sigma_p}\right\} \end{aligned} \tag{A4}$$

The breakeven line is where $S_p^A(w) = S_p$, i.e.

$$S_p(1 - \rho\frac{\sigma}{\sigma_p}) + \frac{r - r_p}{\sigma_p} = 0$$

Rearranging, we obtain

$$r = r_p - S_p(\sigma_p - \rho\sigma) \tag{A5}$$

This is the equation describing the breakeven line.

Accommodating the new asset class

In the argument above, a small weight w of the new asset class is combined with a weight $(1 - w)$ of the original portfolio. In other words, it is assumed that all the existing asset class weights are cut proportionately to accommodate the new class.

We show below that if the current portfolio is efficient, the rate of change of the Sharpe ratio as the new asset class is introduced is not affected by how it is accommodated.

Suppose there are currently n asset classes in the portfolio, with weights $w_1, w_2, ..., w_n$. Denote the new asset class weight by w_0.

Let the changes in portfolio weights be $\Delta w_i, i = 0,...,n$. The weight for the new asset class becomes Δw_0 (remembering that currently $w_0 = 0$) where Δw_0 is small, and the other weights become $w_i + \Delta w_i, i = 1,...,n$. Because the weights must sum to 1, we note that $\sum_{i=1}^{n} \Delta w_i = -\Delta w_0$.

Consider the Sharpe ratio $S_p(w_0, w_1, ..., w_n)$ as a function of the portfolio weights, and write the current value as S_{p0}. Then, expanding S as a multivariate Taylor series,

$$S_p(\Delta w_0, w_1 + \Delta w_1, ..., w_n + \Delta w_n) \approx S_{p0} + \sum_{i-0}^{n} \frac{\partial S_p}{\partial w_i}\Delta w_i \tag{A6}$$

The approximation in (A6) is accurate for small increments Δw_i, $i = 0,...,n$. The partial derivatives $\partial S_p / \partial w_i = 0$, $i = 1,...,n$, otherwise the portfolio could be rearranged to give a higher expected return for the same risk, which is impossible because the portfolio is efficient. Thus

$$S_p(\Delta w_0, w_1 + \Delta w_1, ..., w_n + \Delta w\ ) \approx S_{p0} + \frac{\partial S_p}{\partial w_0} \Delta w_0 \qquad \text{(A7)}$$

independent of the increments $\Delta w_i, i = 1,...,n$. Thus the change in the Sharpe ratio from adding a small weight Δw_0 of the new asset class does not depend on which of the existing asset class weights we choose to reduce to accommodate it.

Finally, comparing (A4) and (A7), we note that $\frac{\partial S_p}{\partial w_0} = S_p(1 - \rho \frac{\sigma}{\sigma_p}) + \frac{r - r_p}{\sigma_p}$.