

Transformation of returns to the time domain: The case of doubling times

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Abstract:

This paper examines the re-expression of returns in the time domain through the analysis of doubling times. Several uses for the doubling time metric are suggested and it is shown that the expected time for an investment to double can be calculated from harmonic means, or by Monte Carlo simulation. Doubling times can be used to provide an alternative calculus for portfolio optimisation. Two alternative risk metrics are suggested for doubling times and the use of either can reproduce the Markowitz efficient frontier.

Keywords: Returns, return transformation, doubling times, portfolio optimisation, risk metrics

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Returns and Doubling Times

1. Introduction

This paper examines the re-expression of returns in the time domain. Traditionally investment performance is measured as the increment in wealth per unit of time. In this paper we turn the measurement round and consider the period required to give a unit increment in wealth, where a unit of wealth is defined to be equal to the initial investment. Thus returns are re-expressed as doubling times.

There are several reasons for the study of doubling times. Doubling times are an intuitively attractive way to express returns. This is evident from the development of rules of thumb for estimating doubling times, such as the rule of seventy-two. Despite this intuitive appeal, there has been little study of doubling times in finance. Transforming returns into the time domain provides a different perspective on returns. Viewing returns from a different perspective may stimulate new ideas and new insights that might not otherwise be obtained. Doubling times may also have advantages over returns in certain applications. We suggest three potential uses for doubling times, in relation to truth in lending, performance measurement, and capital budgeting. We demonstrate a fourth application in portfolio optimisation.

It is a simple matter to compute doubling times period by period. However, the mean of the period by period doubling times (the mathematical expectation) does not give the time over which the investor should expect to double their money (the expected doubling time). We present two approaches to computing the expected doubling time. An analytical approach,

which shows that harmonic means can be used in estimating doubling times. Formulas using harmonic means are given for discrete and continuously compounded rates of return and also for simple interest rates.

A Monte Carlo simulation method is also used to estimate the expected doubling time. The results of the simulation approximate an inverse Gaussian distribution. If returns are normally distributed, then doubling times will follow an inverse Gaussian distribution. Using the properties of the inverse Gaussian distribution we demonstrate that minimising either the skewness, or the inverse of the shape parameter, of the doubling time distribution results in the same results as the standard Markowitz mean variance analysis for returns.

The remainder of this paper is organised as follows. Section 2 outlines the history of doubling times and the possible applications to finance. Section 3 discusses data and methods, while section 4 presents the results. The conclusions of this study are presented in section 5.

2. Doubling Times History, Definition and Uses

For discrete returns the doubling time is approximated by the rule of 72, dividing 72 by the rate of return gives the approximate time for the an investment to double in value. Internet sources often credit Albert Einstein for the rule of 72;¹ but almost 500 years earlier it was mentioned by Luca Pacioli (1494) and quite possibly it predates Pacioli. When compounding is continuous, rather than discrete, the rule of 69 is used instead of the rule of 72.

Although the expression of returns as doubling times has been known for 500 or more years, there has been little academic work on equity market doubling times. Indeed, other

¹ See for example: www.investingpage.com/einsteins-rule-of-72.html

than descriptions of the rules of 72 and 69, there is little on doubling times in the financial literature. However, the concept of doubling times has had more extensive use in other fields. Doubling times are applied to population growth as in Kendall (1949). Doubling times are also used in medicine; a common application is measuring the growth of a tumour, for example Hanks, D'Amico, Epstein and Schultheiss (1993). The converse of doubling times, which are half-lives, are also used in other fields. Half lives are most commonly associated with radioactive decay, but can be applied to anything which decays. Medical applications involve nuclear medicine as in Hendee (1979). The study of population extinctions also makes use of half lives, Brooks, Pimm and Oyugi (1999).

Future values under continuous compounding are given by:

$$FV = PVe^{rt}$$

where FV is the Future Value, PV is the Present Value, r is the rate of return, and t time to maturity.

By setting FV to 2 and PV to 1 and solving for time, t we obtain the doubling time, τ :

$$\tau = \frac{\log(2)}{r} \quad (1)$$

A result which is approximated by the rule of 69 as the natural logarithm of 2 equals 0.6931.

Similarly, future values with discrete returns are given by:

$$FV = PV(1+r)^t$$

Again, by setting FV to 2 and PV to 1 and solving for t we obtain:

$$\tau = \frac{\log(2)}{\log(1+r)} \quad (2)$$

This equation is approximated by the rule of 72.

For simple interest future values are given by:

$$FV = PV(1 + rt)$$

Setting FV equal to 2 and PV equal to 1 and solving for τ we obtain:

$$\tau = \frac{1}{r} \quad (3)$$

We note that, while the formulas above differ, for any given asset they will all give the same doubling time, since no matter how the returns are expressed they all reflect the same underlying investment performance.

We suggest three possible uses for doubling times below and later we show how doubling times can be applied to portfolio optimisation.²

Truth in Lending: Interest rates may expressed as simple or compound rates, compounding may be discrete or continuous, and interest rates may be expressed in nominal or effective terms. Whatever way the interest rate is expressed, there is only one doubling time. Thus doubling times could supply a standard benchmark for comparing loans and would probably have an intuitive appeal to consumers.

Performance Measurement: If doubling times are useful in truth in lending, this also suggests that they might provide a useful way to report performance to investors. For example investment funds could be asked to report how long ago one dollar would need to have been invested with the fund in order to have doubled in value by the current date.

Funds could also report the doubling time based on the current period's return.

² We doubt that this is an exhaustive of the possibilities, but it serves to illustrate the potential usefulness of doubling times.

Capital Budgeting: The payback period continues to be very popular in capital budgeting despite its well known deficiencies, Truong, Partington and Peat (2008). The doubling time probably has the intuitive appeal of payback, but can also give decisions consistent with traditional DCF analysis. This latter would be achieved by setting the required doubling time to match the required rate of return.

3. Data and Method

The computations that underlie the following results were based on continuously compounded returns and hence doubling times were computed using equation (1). As noted above, had we used discrete returns and equation (2) or simple interest rates and equation (3) identical doubling times would have been obtained. A plot of doubling times against continuous returns was generated from equation (1) and is given in Figure 1. It is immediately evident from Figure 1 that negative doubling times (half lives) are a mirror image of positive doubling times. It is also evident that there is a discontinuity at zero. As returns approach zero, negative doubling times tend to minus infinity and positive doubling times tend to plus infinity.

Figure 1: Variation of Doubling Times with Returns

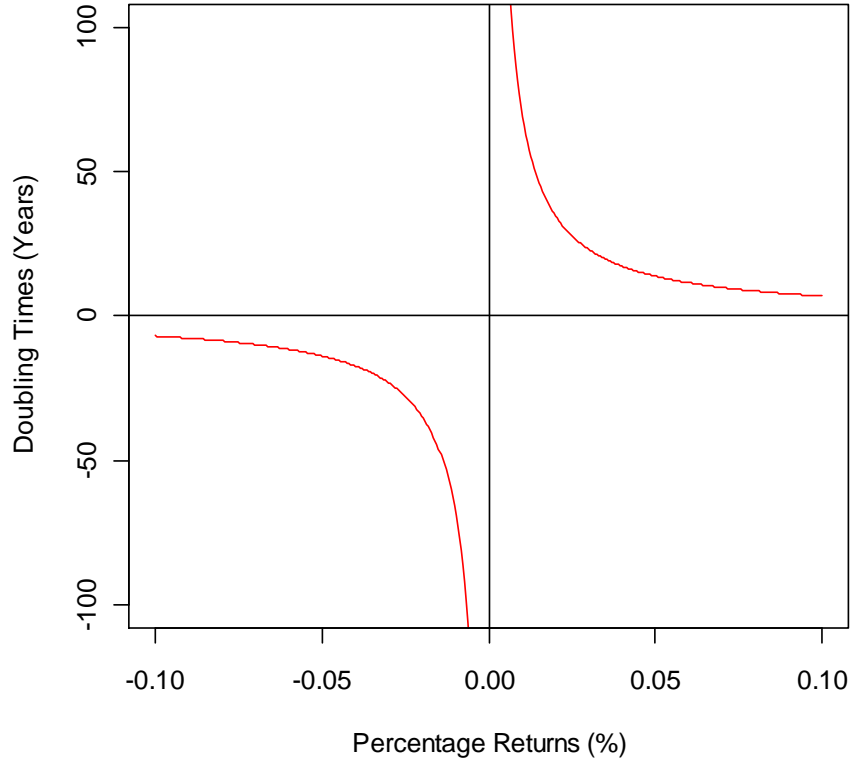


Figure 1 represents a plot of Equation (1) for returns varying from -10% to +10%

Figure 1 suggests that in forming the parameters of doubling time distributions there will be a problem in handling cases with zero returns due to infinite doubling times. Care is also needed in combining doubling times. For example a half life represented by minus five years and a doubling time represented by plus five years have an arithmetic mean of zero years. However an investor who experiences this combination of half-life and doubling time will not instantly double her money. In general the arithmetic mean of doubling times is not the same as the doubling time that investors experience. Investors are naturally

interested in the doubling times they actually experience and we describe below how this parameter can be computed from the time series of doubling times.

One approach to dealing with zero returns and infinite doubling times is to exclude such data from the analysis and report the incidence of zero returns. We do not expect zero returns to be a common problem for actively traded stocks, but it could happen, and is more likely as we move from monthly to daily data. For thinly traded stocks apparent observations of zero return are common. However, it is important to distinguish when zero returns really mean that the price has not changed, and when they mean that a change in price has not been observed because of thin trading. In the latter case the return is not really zero but missing.

Computing expected doubling times

Several approaches can be used in computing the expected doubling time. The simplest and most obvious is to take the compound rate of return (geometric average) over the full set of return observations. Converting that compound return to a doubling time will give the expected doubling time. This approach however, provides no information about the relation between the expected doubling time and the individual doubling times for each observation. We use two approaches to estimating the doubling time directly from the time series of doubling times. One approach is a numerical estimate using Monte Carlo simulation, the other approach is analytical.

In the simulation approach the returns are repeatedly re-sampled at random and a cumulative compound return computed. When this cumulative return sums to two the doubling point has been reached. The number of iterations to reach this point gives the

doubling time and many repetitions of this process give a distribution of possible doubling times about the expected doubling time.

The analytical approach to computing the expected doubling time requires different formulae for discrete and continuous returns. The derivations are as follows,

For continuous compounding:

From equation (1)

$$\tau = \log(2)/r$$

If the rate of return is r_i for a period t_i , the equivalent rate R satisfies:

$$e^{RT} = e^{r_1 t_1 + r_2 t_2 + \dots + r_n t_n}$$

where $T = \sum_i t_i$

Taking the natural logarithm of both sides of the above equation leads to

$$RT = r_1 t_1 + r_2 t_2 + \dots + r_n t_n$$

So R is the weighted arithmetic mean of the r_i . Since $r = \log(2)/\tau$, the above equation can be rewritten as:

$$T \log(2)/\tau = \sum_{i=1}^n t_i \log(2)/\tau_i$$

which simplifies to

$$\tau = T \left(\sum_{i=1}^n t_i / \tau_i \right)^{-1} \quad (4)$$

In this case the expected doubling time is the time weighted harmonic mean of the individual doubling times τ_i .

For discrete compounding:

From equation (2)

$$\tau = \log(2) / \log(1+r)$$

Suppose the returns are (r_1, r_2, \dots, r_n) for n-periods. Then the equivalent rate R satisfies

$$(1+R)^n = \prod_{i=1}^n (1+r_i)$$

So $(1+R)$ is the geometric mean of the one period terms $(1+r_i)$. Since $(1+r)^\tau = 2$, the above equation can be rewritten as:

$$2^{n/\tau} = \prod_{i=1}^n 2^{1/\tau_i}$$

$$2^{n/\tau} = 2^{\sum_{i=1}^n 1/\tau_i}$$

which leads to

$$\tau = n \left(\sum_{i=1}^n \left(\frac{1}{\tau_i} \right) \right)^{-1} \quad (5)$$

This implies that the expected doubling-time is just the harmonic mean of the individual one step doubling times τ_i .

For simple interest rates:

From equation (3)

$$\tau = 1/r$$

Suppose the simple interest rates are r_i for a period t_i . Then the equivalent rate R satisfies

$$R \sum_i^n t_i = \sum_i^n t_i r_i$$

Since $r = 1/\tau$, the above equation can be rewritten as:

$$\frac{1}{\tau} \sum_{i=1}^n t_i = \sum_{i=1}^n t_i / \tau_i$$

which leads to:

$$\tau = T \left(\sum_{i=1}^n \left(\frac{t_i}{\tau_i} \right) \right)^{-1} \quad (6)$$

where $T = \sum_i t_i$.

As for continuous compounding the expected doubling time is the time weighted harmonic mean of the individual doubling times τ_i .

4. Data and Results

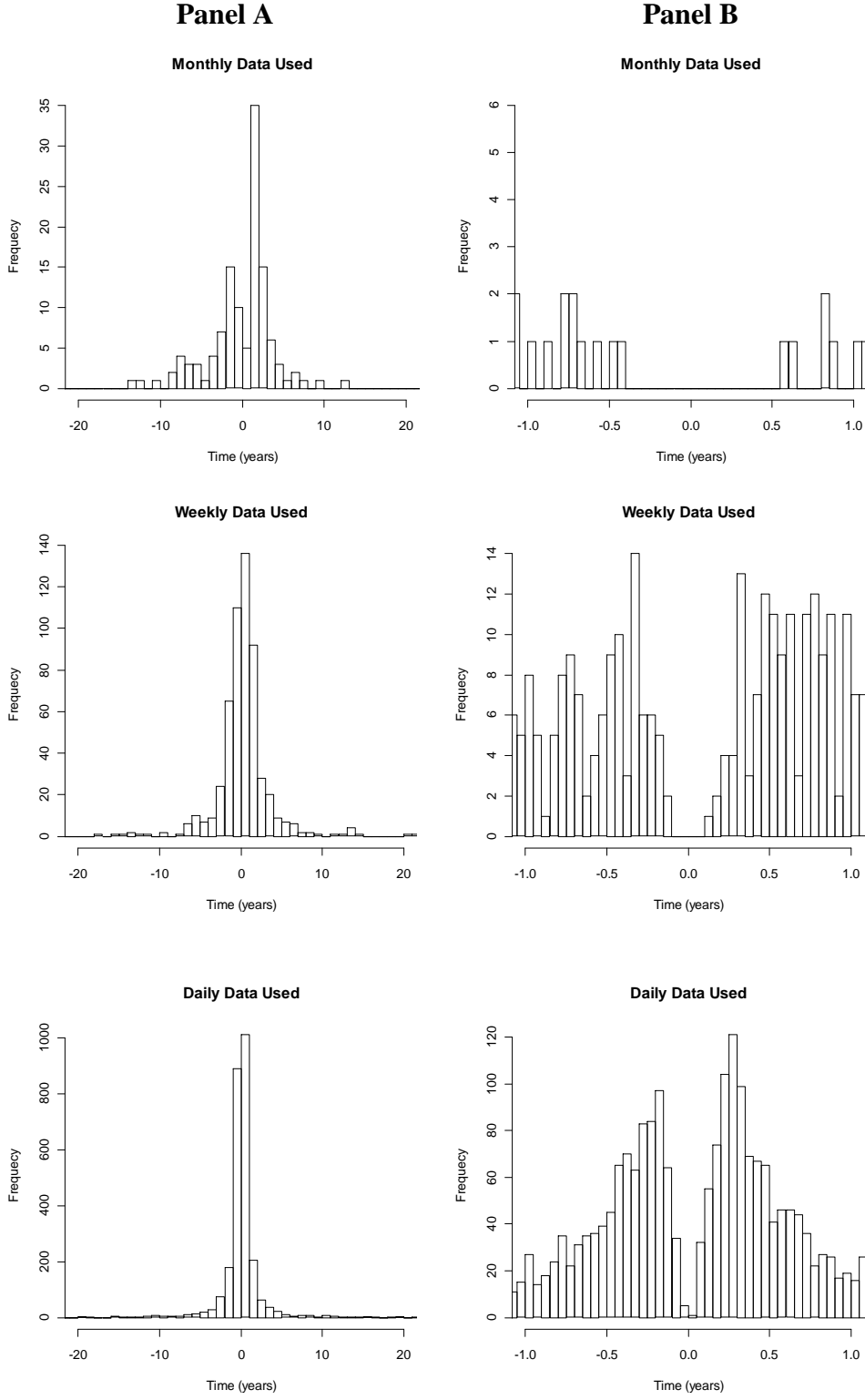
The calculation of doubling times is illustrated with equity data obtained from Bloomberg. The equity data consisted of daily closing prices for the ASX S&P 200 index³ from the 4th of January 2000 to the 15th of September 2010 which gave a total of 2709 data points.

Raw doubling times

Histograms of half lives and doubling times for the ASX S&P 200 index are given in Figure 2 for daily weekly and monthly observations. The great majority of doubling times and half lives are under ten years and it appears that a surprisingly large number of observations at zero. This latter observation, however, is somewhat misleading. Panel B of Figure 2 gives plots drawn from the same distributions, but scaled to magnify the observations centred on zero. Panel B illustrates that there is a discontinuity at zero.

³ As the ASX S&P 200 is a price index it neglects the growth in wealth from the reinvestment of dividends. However, as our purpose is to illustrate the technique for computing expected doubling times, rather than provide a comprehensive picture of the performance of equities, the ASX S&P 200 is sufficient for our purpose.

Figure 2: Histograms of Half Lives and Doubling Times combined for S&P 200 Index.



Histogram of combined doubling times and half lives for the ASX S&P 200 Index using monthly, weekly and daily observations. Panel A shows the full histogram, while Panel B shows a truncated and magnified version of the histogram in order to highlight the discontinuity at zero.

Table 1 provides descriptive statistics for the full data set and for the sub-samples of positive and negative observations, which represent doubling times and half lives respectively. The results are presented at daily, weekly and monthly frequencies. In Panel A the doubling times are reported in years. In Panel B doubling times are reported relative to the frequency of observation, thus the doubling times for daily observations are reported in days and so on. Infinite doubling times are omitted in computing the descriptive statistics, but the number of such instances is reported as the number of zero returns removed.

The problem of computing the expected doubling time as the arithmetic mean of individual doubling times is clearly demonstrated in Table 1. These daily, weekly, and monthly samples are all drawn from the same period (4th of January 2000 to the 15th of September 2010). As these samples all started and finished on the same day, the overall change in wealth is identical regardless of the frequency of measurement. Consequently, the expected doubling time is independent of the frequency of measurement. However, the mean of the doubling times in Table 1 vary with the observation frequency. The mean doubling time based on weekly observations for the full sample is minus 0.0324 years. In contrast, the mean doubling time based on monthly returns for the full sample is 8.032 years. Similar variation is also observed when considering the doubling times and half-lives separately. Investors are naturally interested in the doubling times that they can expect, and that they actually experience, but doubling times that vary with the frequency of return observation will not provide this information.

Table 1: Descriptive Statistics for Doubling Times and Half Lives

Panel A: All values have been reported where periods are measured in years

	Daily Data			Weekly Data			Monthly Data		
	combined	doubling times	half lives	combined	doubling times	half lives	combined	doubling times	half lives
mean	0.084596	2.013169	-2.05609	-0.03241	2.483745	-3.26457	8.032	17.249	-5.011
variance	103.02	131.5834	62.67773	71.28123	19.07258	120.0427	6214.554	10359.59	139.4085
standard deviation	10.14	11.4736	7.91692	8.4428	4.36721	10.9564	78.832	101.7821	11.8071
min	-169.01	0.047367	-169.017	-146.741	0.129703	-146.741	-86.082	0.59799	-86.0828
max	355.1006	355.1006	-0.03063	33.5012	33.5012	-0.1136	877.55	877.5532	-0.42177
median	0.13758	0.5117304	-0.52087	0.455504	1.15171	-1.13576	1.27006	1.92002	-2.04198
zero returns removed	3			1			0		
Data sample size	2708			563			128		

Panel B: All values have been reported corresponding to the interval length of the data used

	Daily Data			Weekly Data			Monthly Data		
	combined	doubling times	half lives	combined	doubling times	half lives	combined	doubling times	half lives
mean	21.99	523.424	-534.58	-1.68538	129.154	-169.75	96.384	206.9944	-60.1402
variance	6964231	8895035	4237015	192744	51572.27	324595	894895	1491781	20074.83
standard deviation	2638.98	2982.45	2058.401	439.027	227.095	569.7328	945.9893	1221.385	141.6857
min	-43944.5	12.31558	-43944.5	-7630.51	6.744558	-7630.51	-1032.99	7.17594	-1032.99
max	92326.16	92326.16	-7.96328	1742.065	1742.06	-5.90723	10530.64	10530.64	-5.06125
median	35.7712	133.049	-135.427	23.6862	59.888	-59.0594	15.24072	23.0403	-24.5038
zero returns removed	3			1			0		

Expected doubling times

Table 2 provides descriptive statistics for the expected doubling time of the ASX S&P 200 Index consistent with an investor's experience. The expected doubling times are derived from the analytical approach and from the simulation using daily, weekly and monthly observations. The expected doubling times are reported both in years and in units of time corresponding to the measurement interval. In all cases, for the analytical calculation, the expected time to double the investment is equal to 18.175 years.

The simulated distributions of doubling times are based on 5000 iterations. The expected doubling time is computed as the arithmetic average of the simulated distribution. The simulated expected doubling times are 18.463, 18.484 and 18.471 years when using the daily, weekly and monthly returns respectively. As the *t*-statistic in Table 2 shows these simulated values are not significantly different to the analytical value of 18.175 years. As subsequently discussed, the appropriate distributional assumption is not normality, but rather that the doubling times follow the inverse Gaussian distribution. Consequently, the *t*-statistic was computed under this assumption following the method of Chhikara and Folks (1989).

Table 2: Descriptive Statistics for the ASX S&P 200 Index Doubling Times

	Daily Data	Weekly Data	Monthly Data
Simulated expected doubling time (Periods)	4651.17	969.81	234.07
Analytical expected doubling time (Periods)	4578.49	953.57	237.38
Simulated standard deviation (Periods)	4674.36	982.37	229.20
Simulated expected doubling time (Years)	18.46	18.48	18.47
Analytical expected doubling time (Years)	18.18	18.18	18.18
Simulated standard deviation (Years)	18.56	18.72	17.92
Critical <i>t</i> statistic	1.96	1.96	1.96
Computed <i>t</i> Statistic	1.08	1.24	1.13

Expected doubling time and variance of doubling times are reported for the simulated doubling times. The analytical doubling times are also reported along with *t*-statistics for the comparison of the analytical doubling time with the simulated doubling time.

The Distribution of Doubling Times

A density plot for the daily, weekly and monthly simulated doubling times is given in Figure 3. Also in Figure 3, the cumulative density function for the simulated distribution is plotted against the cumulative density function for the inverse Gaussian distribution. These P-P plots suggest that the doubling times follow the inverse Gaussian distribution. The intuition for this result is as follows. The inverse Gaussian distribution describes the time it takes to reach a fixed positive level (the first passage time) for a process that follows Brownian Motion with positive drift. This is analogous to the time a stock with positive drift and a stochastic component in returns takes to double its initial value.

Thus far no assumption has been made about the underlying distribution of returns.

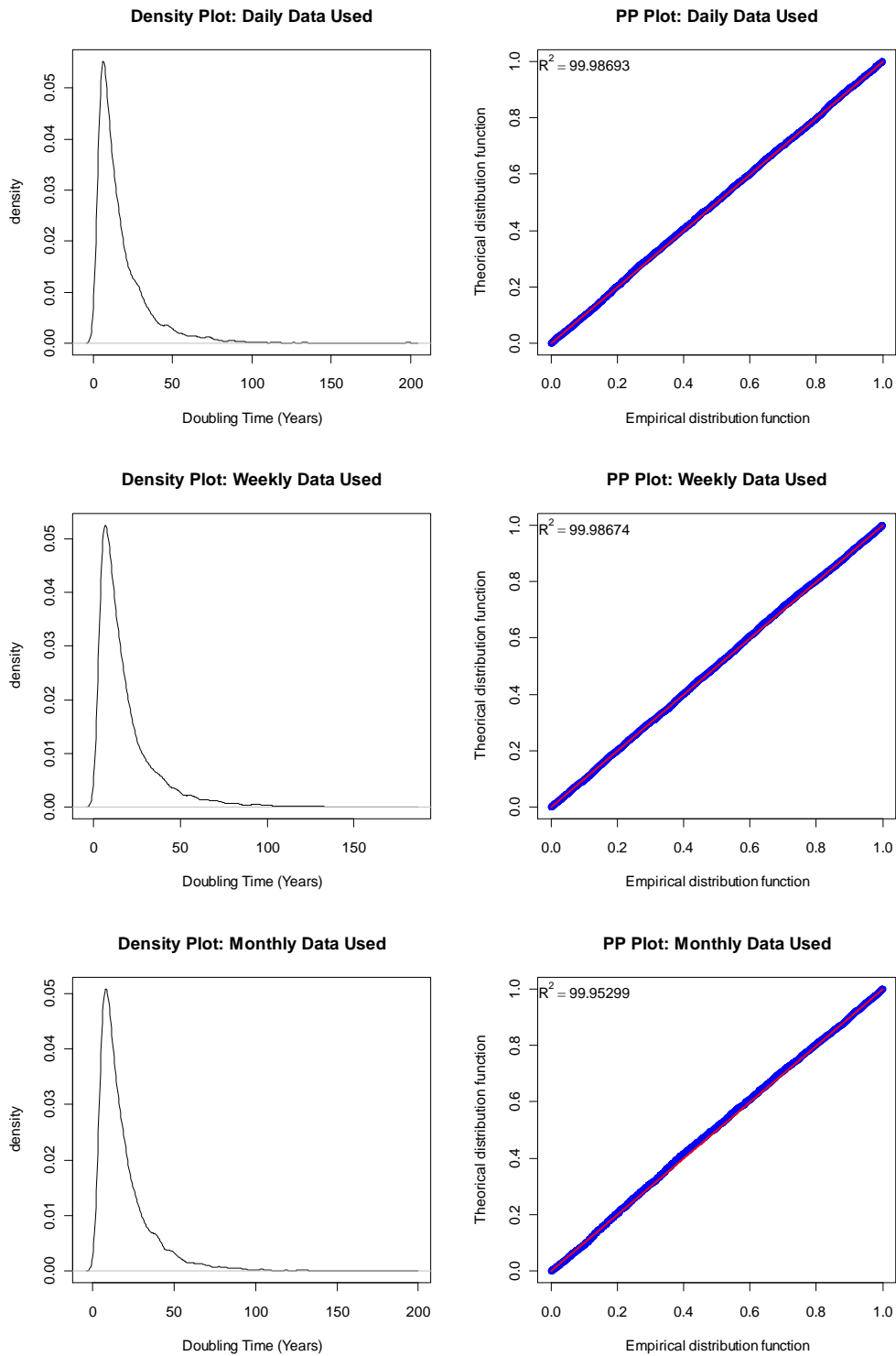
However, if returns are normally distributed then it can be shown that doubling times will follow an inverse Gaussian distribution. Details are given in the appendix, where it is shown that the mean (μ_{DT}) and variance (σ_{DT}^2) of the doubling times are given by:

$$\mu_{DT} = \frac{\log(2)}{\mu_r} \quad (7)$$

$$\sigma_{DT}^2 = \frac{\log(2)\sigma_r^2}{\mu_r^3} \quad (8)$$

Where μ_r and σ_r^2 are the mean and variance of the underlying normally distributed returns.

Figure 3: Doubling time density plot with corresponding P-P plot against an inverse Gaussian distribution.



Traditionally, the inverse Gaussian distribution is not defined by the mean and variance but by the mean and shape parameter. The shape parameter is defined as

$$\lambda = \mu_{DT}^3 / \sigma_{DT}^2 \quad (9)$$

Where μ_{DT} and σ_{DT}^2 are the inverse Gaussian distribution's mean and variance respectively and the DT subscript denotes doubling times.

Using equations (7) and (8) the shape parameter for the doubling time distribution can be written in terms of returns as:

$$\lambda_{DT} = \frac{\log(2)^2}{\sigma_r^2} \quad (10)$$

Thus we can write the distribution of doubling times as:

$$\text{Doubling Times} \sim IG\left(\frac{\log 2}{\mu_r}, \frac{\log(2)^2}{\sigma_r^2}\right) \quad (11)$$

Portfolio Optimisation

In optimising a doubling time portfolio the objective is to minimise the expected doubling time on the assumption that investors wish to amass wealth as quickly as possible, but at the same time investors wish to avoid risk. While minimising doubling time is clearly analogous to maximising returns, the choice of risk metric is less obvious. Either skewness, or the shape parameter, but not variance, can be used in determining the efficient frontier.

We show below that it is possible to obtain the global minimum variance portfolio by minimising the inverse of the shape parameter, subject to portfolio weights summing to one.

We also show that minimising the skewness, subject to portfolio weights summing to one,

gives the tangency portfolio (assuming no risk free asset) of the Markowitz efficient frontier.⁴ By the two fund theorem, the full efficient frontier can be generated as weighted combinations of the global minimum variance and tangency portfolios, Merton (1972). However, we also show below that either skewness, or the shape parameter, can be used to directly generate the full efficient frontier.

From equation (10) the inverse of the shape parameter is:

$$(\lambda_{dr})^{-1} = \frac{\sigma_r^2}{c} \quad \text{where } c = \log(2)^2 \quad (12)$$

Equation (12) shows that the inverse of the shape parameter is the returns variance scaled by a constant. As such, if the inverse of the shape parameter is minimised, with the constraint that portfolio weights sum to one, the resulting portfolio will be the global minimum variance portfolio of returns.

To reduce the risk of waiting for long periods of time to double their investment, investors will desire a reduction in the skewness of doubling times. The skewness, γ , of the inverse Gaussian distribution is defined as:

$$\gamma = 3 \left(\frac{\mu_{DT}}{\lambda_{DT}} \right)^{1/2} \quad (13)$$

Substituting equations (7) and (10) for μ_{DT} and λ_{dr} in equation (13) gives:

$$\gamma = 3 \left(\frac{\sigma_r^2}{\mu_r \log(2)} \right)^{1/2} \quad (14)$$

$$\gamma = c \left(\frac{\sigma_r}{\sqrt{\mu_r}} \right) \quad \text{where } c \text{ is a constant} = \frac{3}{\sqrt{\log(2)}} \quad (15)$$

⁴ In the absence of a risk-free asset this tangency point is the portfolio with the highest ratio of return to risk.

Accordingly, minimising skewness involves minimising a risk to return ratio or, equivalently, maximising a ratio of return to risk.

Using data as given in Broadie (1993), five assets are selected with known means, variances and correlations as given by Tables 3 and 4 below.

Table 3: Asset mean returns and standard deviations and their equivalent doubling times

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Mean (Months)	0.006	0.01	0.014	0.018	0.022
Standard Deviation (Months)	0.085	0.08	0.095	0.09	0.1
Expected Doubling Time (Months)	115.525	69.3147	49.5105	38.5082	31.5067
Doubling Time Standard Deviation (Months)	152.267	66.6044	47.7468	31.0275	25.514

Table 4: Correlation matrix for asset returns

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Asset 1	1	0.3	0.3	0.3	0.3
Asset 2	0.3	1	0.3	0.3	0.3
Asset 3	0.3	0.3	1	0.3	0.3
Asset 4	0.3	0.3	0.3	1	0.3
Asset 5	0.3	0.3	0.3	0.3	1

Given this data solutions to the following constrained optimisations were obtained:

Inverse Shape Parameter.

$$\text{Minimise: } \lambda^{-1} \tag{16}$$

$$\text{Subject to: } \sum w_i = 1$$

Where w_i is the weight of security i in the portfolio. Using equation (12), the optimisation problem defined in equation (16) can be redefined using the percentage returns parameters, or algebraically:

$$\text{Minimise: } \frac{w' \Omega w}{(\log(2))^2}$$

$$\text{Subject to: } w' \iota = 1$$

Where w is the vector of portfolio weights, ι is a vector of ones and Ω is the covariance matrix of the assets in the portfolio

Skewness.

$$\text{Minimise: } \gamma \tag{20}$$

$$\text{Subject to: } \sum w_i = 1$$

Again, the optimisation problem defined in equation (20) can be redefined in terms of percentage returns parameters, by using equation (15):

$$\text{Minimise: } \sqrt[3]{\frac{w' \Omega w}{w' r(\log(2))}}$$

$$\text{Subject to: } w' \iota = 1$$

Again, where w is the vector of portfolio weights, ι is a vector of ones, Ω is the covariance matrix and r is the vector of asset returns.

For these minimisation problems the objective functions will have only one local minimum which corresponds to the global minimum. Accordingly, the weights can be found using the Nelder-Mead method⁵. The asset weightings are computed for each of the optimisation problems and the corresponding portfolio mean and variance are then plotted, along with the Markowitz efficient frontier, in mean-variance space. The results are shown in Figure 4. The small circles represent the results of the above optimisations and these circles plot at the global minimum variance portfolio and the tangency portfolio of the Markowitz efficient set.

⁵ The Nelder-Mead is a downhill Simplex method which although not as efficient as some gradient methods, is more robust.

The next step is to show that the whole Markowitz efficient frontier can be formed using either of the doubling time metrics. To construct the full efficient frontier the objective functions of the optimisations are modified as follows:

$$\begin{aligned} \text{Minimise: } & \Gamma_{\lambda} \times \mu_{DR} + \lambda_{DR}^{-1} & (21) \\ \text{Subject to: } & \sum w_i = 1 \end{aligned}$$

Where Γ is a risk tolerance parameter. In order to trace the efficient frontier the optimisation is undertaken for a range of risk tolerance parameters.

To find the efficient frontier when using skewness as a risk metric, a comparable approach can be taken by solving the following optimisation problem across various risk tolerance levels:

$$\begin{aligned} \text{Minimise: } & \Gamma_{\gamma} \times \mu_{DR} + \gamma & (22) \\ \text{Subject to: } & \sum w_i = 1 \end{aligned}$$

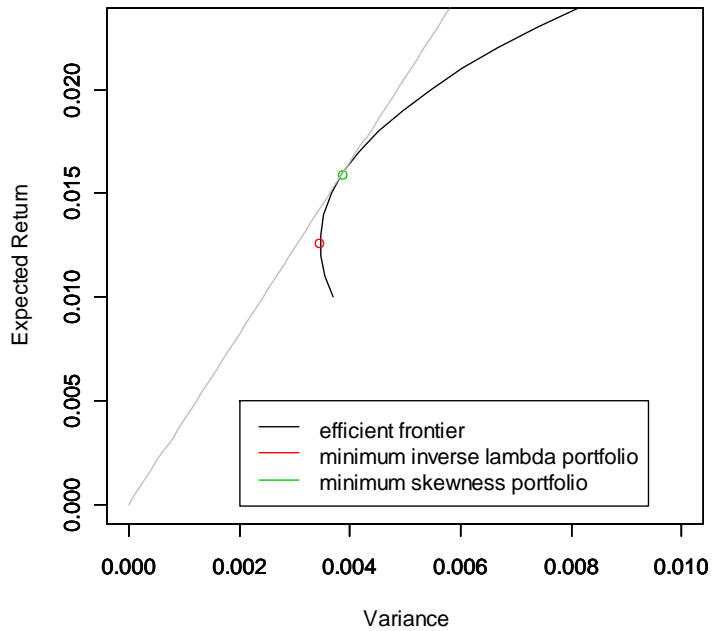
The two frontiers derived from the foregoing optimisations should lie on the Markowitz efficient frontier in the mean-variance plane, as defined by the following objective function:

$$\begin{aligned} \text{Maximise: } & \tau_{\sigma} \times \mu_r - \sigma_r^2 & (23) \\ \text{Subject to: } & \sum w_i = 1 \end{aligned}$$

The efficient frontiers resulting from the three foregoing optimisations are plotted in Figure 4 and match each other exactly.⁶

⁶ While the portfolios along the efficient set are coincident across the three optimisations the values for the risk aversion parameter differs across the optimisations at each point. This is unsurprising as the measures of risk differ. The risk aversion parameters using doubling time metrics are smaller than for the Markowitz analysis.

Figure 4: Efficient frontiers



This figure depicts the efficient frontier using the mean, variance and correlation parameters found in Tables 4 and 5. The efficient frontier was computed using the traditional Markowitz mean-variance framework (equation 20) and is shown as the black line. The optimisations based on the shape parameter and skewness of doubling times (equations 18 and 19 respectively) generate results that exactly match the Markowitz efficient frontier. The red circle represents the portfolio that minimises the inverse shape parameter and which lies at the global minimum variance portfolio. The green circle is the minimum skewness portfolio which is shown to lie on the tangency point between the origin and the frontier (as shown by the grey line).

5. Conclusion

This study examined the properties of doubling times theoretically and by analysis of returns on the ASX. A useful property of doubling times is that no matter how returns are reported there is only one doubling time. It was suggested that doubling times could have applications in truth in lending, in performance measurement and in capital budgeting. Computing doubling times period by period is comparatively simple, but using the time series of the period by period estimates to compute the expected doubling time is a little

more challenging. This requires the use of either harmonic means of the doubling times, or a Monte Carlo simulation.

The results of the Monte Carlo simulation gave a distribution about the expected doubling time that closely approximated the inverse Gaussian distribution. If returns are assumed normal then the doubling time distribution will be inverse Gaussian, with a mean and variance that can be expressed in terms of the mean and variance of the return distribution.

The objective of an investor with respect to doubling times is to minimise the time taken to double their wealth, but of course consideration must be given to risk. Given an inverse Gaussian distribution for doubling times there is a choice of two risk metrics for portfolio optimisation, these are skewness or the shape parameter, but not the variance. Portfolio optimisation can be accomplished in terms of doubling times. Minimising the inverse of the shape parameter gives the global minimum variance portfolio which would be obtained using the classical Markowitz framework. Minimising the skewness of doubling times gives the Markowitz tangency portfolio with the highest reward-to-risk ratio. The skewness result has particular intuitive appeal. This appeal arises as investors would want to minimise positive skew in their investment doubling times. Long waiting periods until their investments double would naturally be undesirable. Furthermore, this doubling time optimisation problem only requires one parameter (skewness) to obtain the tangency portfolio, yet the Markowitz framework requires two parameters (mean and variance), thereby the use of doubling times results in a single parameter yet equivalent optimisation problem. These results provide a new perspective from which to view portfolio theory and an alternative calculus for generating the efficient frontier.

APPENDIX

This appendix demonstrates the relation between the doubling times distribution and the return distribution assuming that the return distribution is normal. The derivation begins by using a transformation documented in Lax (1966), as follows:

The relationship from Lax (1966) shows that if:

$$\frac{dX(t)}{dt} = g(X(t))w(t)$$

Then the transformation

$$Y(X) = \int^X g(X')^{-1} dX'$$

Yields the process satisfying

$$\frac{dY(t)}{dt} = w(t)$$

Now, compounding returns following

$$X(t) = X(0)e^{\int_0^t r(s)ds}$$

give the stochastic differential equation:

$$\frac{dX(t)}{dt} = r(t)X(t)$$

Using the relationship from Lax (1966) defined above, the following is obtained

$$Y(X) = \int^X \frac{dX'}{X'}$$

$$\therefore Y(t) = \log X(t) \tag{A1}$$

Which yields the random process satisfying

$$\frac{dY(t)}{dt} = r(t) \tag{A2}$$

If $E[r(t)dt] = \mu dt$ and $Var[r(t)dt] = \sigma^2 dt$, then this can be written as a stochastic differential equation for a Weiner process with drift μ .

$$dY(t) = \mu dt + \sigma Z(t)\sqrt{dt} \quad (A3)$$

where $Z(t)$ is a Gaussian process with $E[Z(t)] = 0$ and $Var[Z(t)] = 1$

The first passage time of this process is well documented; see Domine (1995), where the expected time to absorption and its variance are defined as:

$$E(T) = \frac{Y_B - Y_0}{\mu}, \quad V(T) = \frac{(Y_B - Y_0)\sigma^2}{\mu^3}$$

Where Y_B is the absorbing barrier and Y_0 is the starting point for the process. This is also known to be an inverse Gaussian distribution

Recalling equation (A1), it is known that $Y(t) = \log X(t)$. So if the starting point $X_0 = 1$, and the absorbing point $X_B = 2$. Then the mean (μ_{DT}) and standard deviation (σ_{DT}^2) of the distribution of doubling times are given by:

$$\mu_{DT} = \frac{\log(2)}{\mu_r} \quad (10)$$

$$\sigma_{DT}^2 = \frac{\log(2)\sigma_r^2}{\mu_r^3} \quad (11)$$

Where μ_r and σ_r^2 are the mean and variance of the underlying normally distributed returns.

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