Consumption-Investment Strategies with Non-Exponential Discounting and Logarithmic Utility

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Abstract

In this paper, we revisit the consumption-investment problem with a general discount function and a logarithmic utility function in a non-Markovian framework. The coefficients in our model, including the interest rate, appreciation rate and volatility of the stock, are assumed to be adapted stochastic processes. Following Yong (2012a,b)'s method, we study an N-person differential game. We adopt a martingale method to solve an optimization problem of each player and characterize their optimal strategies and value functions in terms of the unique solutions of BSDEs. Then by taking limit, we show that a time-consistent equilibrium consumption-investment strategy of the original problem consists of a deterministic function and the ratio of the market price of risk to the volatility, and the corresponding equilibrium value function can be characterized by the unique solution of a family of BSDEs parameterized by a time variable.

Keywords: Consumption-investment problem, Non-exponential discounting, Time-inconsistency, Multi-person differential game, BSDEs

1. Introduction

In recent years, research on the time-inconsistent preferences has attracted an increasing attention. The empirical studies of human behavior reveal that the constant discount rate assumption is unrealistic, (see, for example, Thaler (1981), Ainslie (1992) and Loewenstein & Prelec (1992)). Experimental evidence shows that economic agents are impatient about choices in the short term but are more patient when choosing between long-term alternatives. Particularly, cash flows in the near future tend to be discounted at a significantly higher rate than those occur in the long run. Considering such behavioral feature, economic decisions may be analyzed using the hyperbolic discounting (see Phelps & Pollak (1968)). Indeed, the hyperbolic discounting has been widely adopted in microeconomics, macroeconomics, and behavioral finance, such as Laibson (1997) and Barro (1999) among others.

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However, difficulties arise when we attempt to solve an optimal control problem with a non-constant discount rate by some standardized control techniques, such as dynamic programming approach. In fact, these techniques lead to time inconsistent strategies, i.e, a strategy that is optimal for the initial time may not be optimal later (see, for example, Ekeland & Pirvu (2008) and Yong (2012b)). In other words, the classical dynamic programming principle fails to solve the so-called time-inconsistent control problem. So, how to obtain a time-consistent strategy for time-inconsistent control problems becomes an interesting and challenging problem. In Strotz (1955), the author studies a cake eating problem within a game theoretic framework where the players are the agent and his/her future selves, and seek a subgame perfect Nash equilibrium point for this game. Strotz's work has been pursued by many others, such as Pollak (1968), Peleg & Yaari (1973), Goldman (1980) and Laibson (1997) among others.

Recently, the time inconsistent control problems regain considerable attention in the continuous-time setting. A modified HJB equation is derived in Marín-Solano & Navas (2010) which solves the optimal consumption and investment problem with non-constant discount rate for both naive and sophisticated agents. The similar problem is also considered by another approach in Ekeland & Lazrak (2006) and Ekeland & Pirvu (2008), which provide the precise definition of the equilibrium concept in continuous time for the first time. They characterize the equilibrium policies through the solutions of a flow of BSDEs, and they show, with special form of the discount factor, this BSDE reduces to a system of two ODEs which has a solution. There are some literature following their definition of equilibrium strategy. In Björk & Murgoci (2010), the time-inconsistent control problem is considered in a general Markov framework, and an extended HJB equation together with the verification theorem are derived. Björk et al. (2014) investigates the Markowitz's problem with state-dependent risk aversion by utilizing the extended HJB equation obtained in Björk & Murgoci (2010). Considering the hyperbolic discounting, Ekeland et al. (2012) studies the portfolio management problem for an investor who is allowed to consume and take out life insurance, and they characterize the equilibrium strategy by an integral equation.

Another approach to the time-inconsistent control problem is developed by Yong (2011, 2012a,b). In Yong's papers, a sequence of multi-person hierarchical differential games is studied first and then the time-consistent equilibrium strategy and equilibrium value function are obtained by taking limit. A brief description of the method is given as follows. Let $T > 0$ be the fixed time horizon and $t \in [0, T)$ be the initial time. Take a partition $\Pi = \left\{t_k \,\middle|\, 0 \le k \le N\right\}$ of the time interval $[t, T]$ with $t = t_0 < t_1 < \cdots < t_N = T$, and with the mesh size

$$||\Pi|| = \max_{1 \le k \le N} (t_k - t_{k-1}).$$

Consider an N-person differential game: for $k = 1, 2, \cdots, N$, the k-th player controls the system on $[t_{k-1}, t_k)$, starting from the initial state $(t_{k-1}, X(t_{k-1}))$ which is the terminal state of the $(k-1)$-th player, and tries to maximize his/her own performance functional. Each player knows that the later players will do their best, and will modify their control systems as well as their cost functional. In the performance functional, each player discounts the utility in his/her own way. Then for any given partition Π, a Nash equilibrium strategy is constructed to the corresponding N-person differential game. Finally, it can be shown that as the mesh size $\|\Pi\|$ approaches to zero, the Nash equilibrium strategy to the N-person differential game approaches to the desired time-consistent solution of the original time-inconsistent problem. By this method, Yong (2011, 2012a) considers a deterministic time-inconsistent linear-quadratic control problem. Considering a controlled stochastic differential equation with deterministic coefficients, Yong (2012b) investigates a time-inconsistent problem with a general cost functional and derives an equilibrium HJB equation.

In this paper, we revisit the consumption-investment problem (Merton (1969, 1971)) with a general discount function and a logarithmic utility function. In contrast to the references cited above, we consider this problem in a non-Markovian framework. More specifically, the coefficients in our model, including the interest rate, appreciation rate and volatility of the stock, are assumed to be adapted stochastic processes. To our best knowledge, the literature on the time-inconsistent problem in a non-Markovian model is rather limited. A time-inconsistent stochastic linear-quadratic control problem is studied in a model with random coefficients by Hu et al. (2012). A time-consistent strategy is obtained for the mean-variance portfolio selection by Czichowsky (2013) in a general semimartingale setting. Following Yong's method, we first study an N-person differential game. Similar to Hu et al. (2005) and Cheridito & Hu (2011), we adopt a martingale method to solve an optimization problem of each player and characterize their optimal strategies and value functions in terms of the unique solutions of BSDEs. Then by taking limit, we show that a time-consistent equilibrium consumption-investment strategy of the original problem consists of a deterministic function and the ratio of the market price of risk to the volatility, and the corresponding equilibrium value function can be characterized by the unique solution of a family of BSDEs parameterized by a time variable which can be understood as the initial time of each player in the N-person differential game.

The remainder of this paper is organized as follows. Section 2 introduces the model. In Section 3, we study the N-person differential game. Section 4 gives a time-consistent equilibrium strategy and time-consistent equilibrium value function to the original problem. Section 5 concludes the paper. Some proofs and technical results are collected in the appendices.

2. The Model

Let $T > 0$ be a fixed finite time horizon, and $\{W(t)\}_{0\le t\le T}$ be a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathsf{P})$. Here the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ is the augmentation under P of $\mathcal{F}_t^W := \sigma(W(s), 0 \le s \le t)$, $t \in [0, T]$. We consider a market which consists of a bond and a stock. The price of the bond evolves according to the differential equation

$$\begin{cases} \mathrm{d}B(s) = r(s)B(s)\mathrm{d}s, & s \in [0, T], \\ B(0) = 1. \end{cases}$$

The price of the stock is modelled by the stochastic differential equation

$$\begin{cases} \mathrm{d}S(s) = \mu(s)S(s)\mathrm{d}s + \sigma(s)S(s)\mathrm{d}W(s), & s \in [0,T], \\ S(0) = s_0, \end{cases}$$

where $s_0 > 0$. The interest rate process $\{r(t)\}_{0\le t\le T}$ as well as the appreciation rate $\{\mu(t)\}_{0\le t\le T}$ and volatility $\{\sigma(t)\}_{0\le t\le T}$ of the stock are assumed to be $\{\mathcal{F}_t\}_{0\le t\le T}$-adapted and bounded uniformly in $(t,\omega) \in [0,T]\times\Omega$. In addition, we require that the volatility process $\{\sigma(t)\}_{0\le t\le T}$ is bounded away from zero.

A consumption-investment policy is a bivariate process $(c(t),u(t)) \in \mathbb{R}^+\times\mathbb{R}$, where $c(t)$ is the consumption rate at time t as a proportion of the wealth and $u(t)$ is the proportion of wealth invested in the stock at time t.

Let

$$\begin{aligned} C[t,T] &= \Big\{c : [t,T]\times\Omega \to \mathbb{R}^+ \,\Big|\, c(\cdot) \text{ is a predictable process for which} \\ &\qquad \int_t^T |c(s)|\,\mathrm{d}s < \infty,\ a.s.\Big\}, \\ \mathcal{U}[t,T] &= \Big\{u : [t,T]\times\Omega \to \mathbb{R} \,\Big|\, u(\cdot) \text{ is a predictable process for which} \\ &\qquad \int_t^T |u(s)\sigma(s)|^2\,\mathrm{d}s < \infty,\ a.s.\Big\}. \end{aligned}$$

For any initial time $t \in [0,T]$ and initial wealth $x > 0$, applying a consumption-investment policy $(c(s),u(s)) \in C[t,T]\times\mathcal{U}[t,T]$, the wealth process of the investor, denoted by $X(\cdot)$, is governed by

$$\begin{cases} \mathrm{d}X(s) = [r(s) - c(s) + u(s)\sigma(s)\theta(s)]\,X(s)\mathrm{d}s + u(s)\sigma(s)X(s)\mathrm{d}W(s), & s \in [t,T], \\ X(t) = x, \end{cases} \tag{1}$$

where

$$\theta(s) := \frac{\mu(s) - r(s)}{\sigma(s)}.$$

To emphasize the dependence of the wealth process on the initial state and the policy, we also write the solution of (1) as $X(\cdot;t,x,c(\cdot),u(\cdot))$.

In this paper, we focus on the logarithmic utility function. At any initial time $t \in [0,T]$ with initial wealth $x > 0$, the performance functional, i.e. the expected discounted utility from the consumption and terminal wealth is given by

$$J(t,x;c(\cdot),u(\cdot)) = \mathsf{E}_t\left[\int_t^T h(s-t)\ln(c(s)X(s))\mathrm{d}s + h(T-t)\ln X(T)\right],$$

where $\mathsf{E}_t[\cdot] = \mathsf{E}[\cdot|\mathcal{F}_t]$ and $h(\cdot)$ is a general discount function satisfying

$$h(0) = 1, \quad h(\cdot) > 0, \quad h'(\cdot) \leq 0, \quad \int_0^T h(s)\mathrm{d}s < \infty.$$

We also impose a technical assumption on h.

Assumption 1. *There exists a constant $C > 0$ such that $|h(s) - h(t)| \leq C|s-t|$, for all $s,t \in [0,T]$.*

Note that Assumption 1 is satisfied by many discount functions, such as exponential discount functions, mixture of exponential functions and hyperbolic discount functions.

Definition 2. Given an initial state $(t,x) \in [0,T] \times (0,\infty)$, a strategy $(c(\cdot),u(\cdot)) \in C[t,T] \times \mathcal{U}[t,T]$ is said to be admissible if

$$\mathsf{E}_t\left[\int_t^T |\ln(c(s)X(s))|\,\mathrm{d}s\right] < \infty.$$

We denote by $\mathcal{A}(t,x)$ the class of all such admissible strategies.

Since we understand $\ln x$ to be $-\infty$ for $x \leq 0$ by convention, an admissible strategy satisfies $c(\cdot) > 0$, $\mathrm{d}t \times \mathsf{P}$-a.e., where $\mathrm{d}t$ is the Lebesgue measure on $[0,T]$.

Problem (N)**.** For given $(t,x) \in [0,T] \times (0,\infty)$, find $(\hat{c}(\cdot),\hat{u}(\cdot)) \in \mathcal{A}(t,x)$ such that

$$J(t,x;\hat{c}(\cdot),\hat{u}(\cdot)) = \sup_{(c(\cdot),u(\cdot)) \in \mathcal{A}(t,x)} J(t,x;c(\cdot),u(\cdot)). \tag{2}$$

It is well-known that Problem (N) is time-inconsistent, i.e., if we find some $(\hat{c}(\cdot),\hat{u}(\cdot)) \in \mathcal{A}(t,x)$ such that (2) is satisfied for initial state (t,x), we do not have

$$J(\tau,X(\tau;t,x,\hat{c}(\cdot),\hat{u}(\cdot));(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[\tau,T]}) = \sup_{(c(\cdot),u(\cdot)) \in \mathcal{A}(\tau,X(\tau;t,x,\hat{c}(\cdot),\hat{u}(\cdot)))} J(\tau,X(\tau;t,x,\hat{c}(\cdot),\hat{u}(\cdot);c(\cdot),u(\cdot))$$

in general, for any $\tau \in [t,T]$, where $(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[\tau,T]}$ is the restriction of $(\hat{c}(\cdot),\hat{u}(\cdot))$ on the time interval $[\tau,T]$. We refer the readers to Ekeland & Pirvu (2008) and Yong (2012b) for examples of time-inconsistent consumption-investment problem with more details.

We end this section by introducing two notations. For any $0 \leq T_1 < T_2 \leq T$, we denote by $\mathbb{H}^2_{T_1,T_2}(\mathbb{R})$ the space of all predictable processes $\phi : \Omega \times [T_1,T_2] \to \mathbb{R}$ such that $\|\phi\|^2 := \mathsf{E}_{T_1}\left[\int_{T_1}^{T_2} |\phi(t)|^2\,\mathrm{d}t\right] < +\infty$, and by $\mathbb{H}^\infty_{T_1,T_2}(\mathbb{R})$ the space of all essentially bounded predictable processes $Y : \Omega \times [T_1,T_2] \to \mathbb{R}$.

3. Multi-Person Differential Game

Let $t \in [0,T)$ and $\mathcal{P}[t,T]$ be the set of all partitions $\Pi = \left\{t_k \,\middle|\, 0 \leq k \leq N\right\}$ of $[t,T]$ with $t = t_0 < t_1 < \cdots < t_N = T$. The mesh size of Π is

$$\|\Pi\| = \max_{1 \leq k \leq N}(t_k - t_{k-1}).$$

Given the initial state $(t,x)\in[0,T]\times(0,\infty)$, we consider the following N-person differential game associated with partition $\Pi\in\mathcal{P}[t,T]$.

Let us start with Player (N) who makes the consumption-investment strategy on $[t_{N-1},t_N]$. For $x_{N-1}\in(0,\infty)$, consider the following wealth process

$$\begin{cases}\mathrm{d}X_N(s)=[r(s)-c_N(s)+\theta(s)\sigma(s)u_N(s)]\,X_N(s)\mathrm{d}s+\sigma(s)u_N(s)X_N(s)\mathrm{d}W(s), \quad s\in[t_{N-1},t_N],\\ X_N(t_{N-1})=x_{N-1},\end{cases} \tag{3}$$

and the performance functional

$$J_N(t_{N-1},x_{N-1};c_N(\cdot),u_N(\cdot))=\mathsf{E}_{t_{N-1}}\left[\int_{t_{N-1}}^{t_N}h(s-t_{N-1})\ln(c_N(s)X_N(s))\,\mathrm{d}s+h(t_N-t_{N-1})\ln X_N(t_N)\right]. \tag{4}$$

Note that,

$$J_N(t_{N-1},x_{N-1};c_N(\cdot),u_N(\cdot))=J(t_{N-1},x_{N-1};c_N(\cdot),u_N(\cdot)),$$

for $x_{N-1}>0$.

Definition 3. A consumption-investment strategy $(c_N(\cdot),u_N(\cdot))\in C[t_{N-1},t_N]\times\mathcal{U}[t_{N-1},t_N]$ is said to be admissible for Player (N) with initial state $x_{N-1}\in(0,\infty)$, if

$$\mathsf{E}_{t_{N-1}}\left[\int_{t_{N-1}}^{t_N}|\ln(c_N(s)X_N(s))|\,\mathrm{d}s\right]<\infty.$$

We denote by $\mathcal{A}_N(t_{N-1},x_{N-1})$ the class of all such admissible strategies.

Problem (C_N)**.** For any $x_{N-1}\in(0,\infty)$, find a strategy $(\hat{c}_N(\cdot),\hat{u}_N(\cdot))\in\mathcal{A}_N(t_{N-1},x_{N-1})$ such that

$$\begin{aligned} J_N(t_{N-1},x_{N-1};\hat{c}_N(\cdot),\hat{u}_N(\cdot)) &= V_\Pi(t_{N-1},x_{N-1})\\ &:= \sup_{(\hat{c}_N(\cdot),\hat{u}_N(\cdot))\in\mathcal{A}_N(t_{N-1},x_{N-1})} J_N(t_{N-1},x_{N-1};c_N(\cdot),u_N(\cdot)). \end{aligned} \tag{5}$$

Theorem A.3 shows the value function $V_\Pi(t_{N-1},x_{N-1})$, the optimal strategy $(\hat{c}_N(\cdot),\hat{u}_N(\cdot))$ and the optimal wealth process $\hat{X}_N(\cdot)\equiv\hat{X}_N(\cdot;t_{N-1},x_{N-1})$.

Next, we consider an optimal control problem for Player $(N-1)$ on $[t_{N-2},t_{N-1})$. For each $x_{N-2}\in(0,\infty)$, consider the following wealth process

$$\begin{cases}\mathrm{d}X_{N-1}(s)=[r(s)-c_{N-1}(s)+\theta(s)\sigma(s)u_{N-1}(s)]\,X_{N-1}(s)\mathrm{d}s+\sigma(s)u_{N-1}(s)X_{N-1}(s)\mathrm{d}W(s), \quad s\in[t_{N-2},t_{N-1}),\\ X_{N-1}(t_{N-2})=x_{N-2}.\end{cases}$$

Recall that Player $(N-1)$ can only control the system on $[t_{N-2},t_{N-1})$ and Player (N) will take over at t_{N-1} to control the system thereafter. Moreover, Player $(N-1)$ knows that Player (N) will play optimally based on the initial pair $(t_{N-1},X_{N-1}(t_{N-1}))$, which is the terminal pair of Player $(N-1)$.

Hence, the performance functional of Player $(N-1)$ should be

$$
\begin{aligned}
J_{N-1}(t_{N-2},x_{N-2};c_{N-1}(\cdot),u_{N-1}(\cdot)) \;=\; & \mathsf{E}_{t_{N-2}}\Big[\int_{t_{N-2}}^{t_{N-1}} h(s-t_{N-2})\ln(c_{N-1}(s)X_{N-1}(s))\,\mathrm{d}s \\
& +\int_{t_{N-1}}^{t_N} h(s-t_{N-2})\ln\Big(\hat{c}_N(s)\hat{X}_N(s;t_{N-1},X_{N-1}(t_{N-1}))\Big)\mathrm{d}s \\
& +h(t_N-t_{N-2})\ln\hat{X}_N(t_N;t_{N-1},X_{N-1}(t_{N-1}))\Big]. \qquad (6)
\end{aligned}
$$

Note that in (6) Player $(N-1)$ "discounts" the future utility in his/her own way, i.e., he/she uses the discount function $h(s-t_{N-2})$ for $s\in[t_{N-2},t_N]$.

Let $g_{N-1}(\cdot)$ be a bounded positive function defined on $[t_{N-2},t_N]$ by the ODE

$$
\begin{cases}
g'_{N-1}(s) = -g_{N-1}(s)\frac{h'(s-t_{N-2})}{h(s-t_{N-2})} - 1, \quad s\in[t_{N-2},t_N],\\
g_{N-1}(t_N) = 1.
\end{cases}
$$

and $(\mathcal{Y}_N(\cdot),\mathcal{Z}_N(\cdot))\in\mathbb{H}^{\infty}_{t_{N-1},t_N}(\mathbb{R})\times\mathbb{H}^{2}_{t_{N-1},t_N}(\mathbb{R})$ be the unique solution of the BSDE

$$
\mathcal{Y}_N(\tau) = \int_\tau^{t_N}\mathcal{F}_N(s,\mathcal{Y}_N(s))\mathrm{d}s + \int_\tau^{t_N}\mathcal{Z}_N(s)\mathrm{d}W(s), \quad \tau\in[t_{N-1},t_N],
$$

where

$$
\mathcal{F}_N(s,y) = -\frac{1}{g_{N-1}(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_{N-1}(s)}\ln g_N(s) + \frac{1}{g_N(s)} - r(s).
$$

From Proposition A.4, we have

$$
\begin{aligned}
& J_{N-1}(t_{N-2},x_{N-2};c_{N-1}(\cdot),u_{N-1}(\cdot)) \\
=& \mathsf{E}_{t_{N-2}}\Big[\int_{t_{N-2}}^{t_{N-1}} h(s-t_{N-2})\ln(c_{N-1}(s)X_{N-1}(s))\,\mathrm{d}s \\
& +h(t_{N-1}-t_{N-2})g_{N-1}(t_{N-1})\left[\ln X_{N-1}(t_{N-1})-\mathcal{Y}_N(t_{N-1})\right]\Big].
\end{aligned}
$$

Definition 4. A consumption-investment strategy $(c_{N-1}(\cdot),u_{N-1}(\cdot))\in C[t_{N-2},t_{N-1}]\times\mathcal{U}[t_{N-2},t_{N-1}]$ is said to be admissible for Player $(N-1)$ with initial state $x_{N-2}\in(0,\infty)$, if

$$
\mathsf{E}_{t_{N-2}}\left[\int_{t_{N-2}}^{t_{N-1}}|\ln(c_{N-1}(s)X_{N-1}(s))|\,\mathrm{d}s\right]<\infty.
$$

We denote by $\mathcal{A}_{N-1}(t_{N-2},x_{N-2})$ the class of all such admissible strategies.

Problem (C_{N-1})**.** For any $x_{N-2}\in(0,\infty)$, find a strategy $(\hat{c}_{N-1}(\cdot),\hat{u}_{N-1}(\cdot))\in\mathcal{A}_{N-1}(t_{N-2},x_{N-2})$ such that

$$
\begin{aligned}
J_{N-1}(t_{N-2},x_{N-2};\hat{c}_{N-1}(\cdot),\hat{u}_{N-1}(\cdot)) \;=\; & V_\Pi(t_{N-2},x_{N-2}) \\
:= & \sup_{(c_{N-1}(\cdot),u_{N-1}(\cdot))\in\mathcal{A}_{N-1}(t_{N-2},x_{N-2})} J_{N-1}(t_{N-2},x_{N-2};c_{N-1}(\cdot),u_{N-1}(\cdot)).
\end{aligned}
$$

In Theorem A.5, we get the value function $V_\Pi(t_{N-2},x_{N-2})$, the optimal strategy $(\hat{c}_{N-1}(\cdot),\hat{u}_{N-1}(\cdot))$ and the optimal wealth process $\hat{X}_{N-1}(\cdot)\equiv\hat{X}_{N-1}(\cdot;t_{N-2},x_{N-2})$.

Let

$$(\bar{c}_{N-1}(s),\bar{u}_{N-1}(s))=\begin{cases}(\hat{c}_{N-1}(s),\hat{u}_{N-1}(s)), & s\in[t_{N-2},t_{N-1}),\\ (\hat{c}_N(s),\hat{u}_N(s)), & s\in[t_{N-1},t_N],\end{cases}$$

and $\bar{X}_{N-1}(s)\equiv\bar{X}_{N-1}(s;t_{N-2},x_{N-2})$, $s\in[t_{N-2},t_N]$ be the unique solution to the SDE

$$\begin{cases}\mathrm{d}\bar{X}_{N-1}(s)=[r(s)-\bar{c}_{N-1}(s)+\theta(s)\sigma(s)\bar{u}_{N-1}(s)(s)]\bar{X}_{N-1}(s)\mathrm{d}s+\sigma(s)\bar{u}_{N-1}(s)\bar{X}_{N-1}(s)\mathrm{d}W(s), & s\in[t_{N-2},t_N],\\ \bar{X}_{N-1}(t_{N-2})=x_{N-2}.\end{cases}\tag{7}$$

Similarly, we can state an optimal control problem for Player $(N-2)$ on $[t_{N-3},t_{N-2})$. For each $x_{N-3}\in(0,\infty)$, consider the following wealth process

$$\begin{cases}\mathrm{d}X_{N-2}(s)=[r(s)-c_{N-2}(s)+\theta(s)\sigma(s)u_{N-2}(s)]X_{N-2}(s)\mathrm{d}s+\sigma(s)u_{N-2}(s)X_{N-2}(s)\mathrm{d}W(s), & s\in[t_{N-3},t_{N-2}],\\ X_{N-2}(t_{N-3})=x_{N-3}.\end{cases}$$

The performance functional of Player $(N-2)$ should be

$$\begin{aligned}J_{N-2}(t_{N-3},x_{N-3};c_{N-2}(\cdot),u_{N-2}(\cdot)) \;=\; & \mathsf{E}_{t_{N-3}}\Big[\int_{t_{N-3}}^{t_{N-2}} h(s-t_{N-3})\ln(c_{N-2}(s)X_{N-2}(s))\,\mathrm{d}s\\ &+\int_{t_{N-2}}^{t_N} h(s-t_{N-3})\ln\Big(\bar{c}_{N-1}(s)\bar{X}_{N-1}(s;t_{N-2},X_{N-2}(t_{N-2}))\Big)\mathrm{d}s\\ &+h(t_N-t_{N-3})\ln\bar{X}_{N-1}(t_N;t_{N-2},X_{N-2}(t_{N-2}))\Big],\end{aligned}$$

where $\bar{X}_{N-1}(\cdot;t_{N-2},X_{N-2}(t_{N-2}))$ is the solution to (7) with initial state $(t_{N-2},X_{N-2}(t_{N-2}))$.

Let $g_{N-2}(\cdot)$ be a bounded positive function defined on $[t_{N-3},t_N]$ by the ODE

$$\begin{cases}g'_{N-2}(s)=-g_{N-2}(s)\frac{h'(s-t_{N-3})}{h(s-t_{N-3})}-1, & s\in[t_{N-3},t_N],\\ g_{N-2}(t_N)=1,\end{cases}$$

and $(\mathcal{Y}_{N-1}(\cdot),\mathcal{Z}_{N-1}(\cdot))\in\mathbb{H}^\infty_{t_{N-2},t_{N-1}}(\mathbb{R})\times\mathbb{H}^2_{t_{N-2},t_{N-1}}(\mathbb{R})$ be the unique solution of the BSDE

$$\mathcal{Y}_{N-1}(\tau)=\int_\tau^{t_N}\mathcal{F}_{N-1}(s,\mathcal{Y}_{N-1}(s))\mathrm{d}s+\int_\tau^{t_N}\mathcal{Z}_{N-1}(s)\mathrm{d}W(s),\quad \tau\in[t_{N-2},t_N],$$

where

$$\mathcal{F}_{N-1}(s,y)=-\frac{1}{g_{N-2}(s)}y-\frac{1}{2}\theta^2(s)+\frac{1}{g_{N-2}(s)}\sum_{k=N-1}^{N}\ln g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s)+\sum_{k=N-1}^{N}\frac{1}{g_k(s)}\mathbf{1}_{[t_{k-1},t_k)}(s)-r(s).$$

It follows from Proposition A.6 that

$$\begin{aligned}
&J_{N-2}\left(t_{N-3}, x_{N-3}; c_{N-2}(\cdot), u_{N-2}(\cdot)\right) \\
=&\mathsf{E}_{t_{N-3}}\left[\int_{t_{N-3}}^{t_{N-2}} h(s-t_{N-3})\ln\left(c_{N-2}(s)X_{N-2}(s)\right)\mathrm{d}s\right. \\
&\left.+h(t_{N-2}-t_{N-3})g_{N-2}(t_{N-2})\left[\ln X_{N-2}(t_{N-2})-\mathcal{Y}_{N-1}(t_{N-2})\right]\right].
\end{aligned}$$

Definition 5. A consumption-investment strategy $(c_{N-2}(\cdot), u_{N-2}(\cdot)) \in C[t_{N-3}, t_{N-2}] \times \mathcal{U}[t_{N-3}, t_{N-2}]$ is said to be admissible for Player $(N-2)$ with initial state $x_{N-3} \in (0,\infty)$, if

$$\mathsf{E}_{t_{N-3}}\left[\int_{t_{N-3}}^{t_{N-2}} \left|\ln\left(c_{N-2}(s)X_{N-2}(s)\right)\right|\mathrm{d}s\right] < \infty.$$

We denote by $\mathcal{A}_{N-2}(t_{N-3}, x_{N-3})$ the class of all such admissible strategies.

Problem (C_{N-2})**.** For any $x_{N-3} \in (0,\infty)$, find a strategy $(\hat{c}_{N-2}(\cdot), \hat{u}_{N-2}(\cdot)) \in \mathcal{A}_{N-2}(t_{N-3}, x_{N-3})$ such that

$$\begin{aligned}
J_{N-2}\left(t_{N-3}, x_{N-3}; \hat{c}_{N-2}(\cdot), \hat{u}_{N-2}(\cdot)\right) &= V_{\Pi}(t_{N-3}, x_{N-3}) \\
&:= \sup_{(c_{N-2}(\cdot), u_{N-2}(\cdot)) \in \mathcal{A}_{N-2}(t_{N-3}, x_{N-3})} J_{N-2}\left(t_{N-3}, x_{N-3}; c_{N-2}(\cdot), u_{N-2}(\cdot)\right).
\end{aligned}$$

The value function $V_{\Pi}(t_{N-3}, x_{N-3})$ and the optimal strategy $(\hat{c}_{N-2}(\cdot), \hat{u}_{N-2}(\cdot))$ are given by Theorem A.7,

The above procedure can be continued recursively. For $1 \le k \le N$, we define a bounded positive function $g_k(\cdot)$ on $[t_{k-1}, t_N]$ by the ODE

$$\begin{cases} g_k'(s) = -g_k(s)\frac{h'(s-t_{k-1})}{h(s-t_{k-1})} - 1, \quad s \in [t_{k-1}, t_N], \\ g_k(t_N) = 1. \end{cases}$$

For $1 \le k \le N$, let $(\mathcal{Y}_k(\cdot), \mathcal{Z}_k(\cdot)) \in \mathbb{H}^{\infty}_{t_{k-1},t_k}(\mathbb{R}) \times \mathbb{H}^{2}_{t_{k-1},t_k}(\mathbb{R})$ be the unique solution of the BSDE

$$\mathcal{Y}_k(\tau) = \int_{\tau}^{t_N} \mathcal{F}_k(s, \mathcal{Y}_k(s))\mathrm{d}s + \int_{\tau}^{t_N} \mathcal{Z}_k(s)\mathrm{d}W(s), \quad \tau \in [t_{k-1}, t_N],$$

with the driver

$$\mathcal{F}_k(s,y) = -\frac{1}{g_{k-1}(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_{k-1}(s)}\sum_{n=k}^{N}\ln g_n(s)\mathbf{1}_{[t_{n-1},t_n)}(s) + \sum_{n=k}^{N}\frac{1}{g_n(s)}\mathbf{1}_{[t_{n-1},t_n)}(s) - r(s).$$

Recall that the optimal problem of Player (N) is described by (3-5). By induction, we know that for Player $(k), 1 \le k \le N-1$, with initial state $x_{k-1} \in (0,\infty)$, the state process is

$$\begin{cases} \mathrm{d}X_k(s) = [r(s) - c_k(s) + \theta(s)\sigma(s)u_k(s)]\, X_k(s)\mathrm{d}s + \sigma(s)u_k(s)X_k(s)\mathrm{d}W(s), \quad s \in [t_{k-1}, t_k), \\ X_k(t_{k-1}) = x_{k-1}, \end{cases}$$

and the performance functional is

$$\begin{aligned} J_k\left(t_{k-1}, x_{k-1}; c_k(\cdot), u_k(\cdot)\right) \;=\; & \mathsf{E}_{t_{k-1}}\Bigg[\int_{t_{k-1}}^{t_k} h(s - t_{k-1}) \ln\left(c_k(s)X_k(s)\right) \mathrm{d}s \\ & + \int_{t_k}^{t_N} h(s-t_{k-1}) \ln\Big(\bar{c}_{k+1}(s)\bar{X}_{k+1}(s; t_k, X_k(t_k))\Big)\mathrm{d}s \\ & + h(t_N - t_{k-1}) \ln \bar{X}_{k+1}(t_N; t_k, X_k(t_k))\Bigg], \end{aligned}$$

where

$$(\bar{c}_{k+1}(s), \bar{u}_{k+1}(s)) = \sum_{i=k+1}^{N} (\hat{c}_i(s), \hat{u}_i(s))\, \mathbf{1}_{[t_{i-1}, t_i)}(s),$$

and $\bar{X}_{k+1}(\cdot; t_k, x_k)$ is the unique solution of the SDE

$$\begin{cases} \mathrm{d}\bar{X}_{k+1}(s) = [r(s) - c_{k+1}(s) + \theta(s)\sigma(s)\bar{u}_{k+1}(s)]\, \bar{X}_{k+1}(s)\mathrm{d}s + \sigma(s)\bar{u}_{k+1}\bar{X}_{k+1}(s)(s)\mathrm{d}W(s), \quad s \in [t_k, t_N], \\ \bar{X}_{k+1}(t_k) = x_k. \end{cases}$$

Let $\mathcal{Y}_{N+1}(t_N) := 0$. It follows form Proposition A.8 that the performance functional of Player $(k), 1 \le k \le N$, is given by

$$J_k\left(t_{k-1}, x_{k-1}; c_k(\cdot), u_k(\cdot)\right) = \mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} h(s-t_{k-1}) \ln\left(c_k(s)X_k(s)\right)\mathrm{d}s + h(t_k - t_{k-1}) g_k(t_k) \left[\ln X_k(t_k) - \mathcal{Y}_{k+1}(t_k)\right]\right]. \tag{8}$$

Definition 6. For $k = 1, 2\cdots, N$, a consumption-investment strategy $(c_k(\cdot), u_k(\cdot)) \in C[t_{k-1}, t_k] \times \mathcal{U}[t_{k-1}, t_k]$ is said to be admissible for Player (k) with initial state $x_{k-1} \in (0,\infty)$, if

$$\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} |\ln\left(c_k(s)X_k(s)\right)| \mathrm{d}s\right] < \infty.$$

We denote by $\mathcal{A}_k(t_{k-1}, x_{k-1})$ the class of all such admissible strategies.

Problem (C_k). For $k = 1, 2\cdots, N$, for any $x_{k-1} \in (0,\infty)$, find a strategy $(\hat{c}_k(\cdot), \hat{u}_k(\cdot)) \in \mathcal{A}_k(t_{k-1}, x_{k-1})$ such that

$$\begin{aligned} J_k\left(t_{k-1}, x_{k-1}; \hat{c}_k(\cdot), \hat{u}_k(\cdot)\right) \;=\; & V_\Pi(t_{k-1}, x_{k-1}) \\ :=\; & \sup_{(c_k(\cdot), u_k(\cdot)) \in \mathcal{A}_k(t_{k-1}, x_{k-1})} J_k\left(t_{k-1}, x_{k-1}; c_k(\cdot), u_k(\cdot)\right). \end{aligned}$$

Noting that (8) is in the form of (A.2), Problem (C_k) is a special case of Problem (C) studied in

Appendix A.1. From Theorem A.2, we have the following solution to the Problem (C_k).

Theorem 7. *For $k=1,2\cdots,N$, the value function of Problem (C_k) is given by*

$$V_\Pi(t_{k-1},x_{k-1})=g_k(t_{k-1})\left[\ln x_{k-1}-Y_k(t_{k-1})\right],\quad x_{k-1}(0,\infty),$$

where $Y_k(\cdot)$ is defined by the unique solution $(Y_k(\cdot),Z_k(\cdot))\in\mathbb{H}^\infty_{t_{k-1},t_k}(\mathbb{R})\times\mathbb{H}^2_{t_{k-1},t_k}(\mathbb{R})$ of the BSDE

$$Y_k(\tau)=\mathcal{Y}_{k+1}(t_k)+\int_\tau^{t_k}f_k(s,Y_k(s))\mathrm{d}s+\int_\tau^{t_k}Z_k(s)\mathrm{d}W(s),\quad \tau\in[t_{k-1},t_k],$$

with the driver

$$f_k(s,y)=-\frac{1}{g_k(s)}y-\frac{1}{2}\theta^2(s)+\frac{1}{g_k(s)}(\ln g_k(s)+1)-r(s).$$

The optimal trading strategy $(\hat c_k(\cdot),\hat u_k(\cdot))\in\mathcal{A}_k(t_{k-1},x_{k-1})$ is given by

$$\hat c_k(s)=\frac{1}{g_k(s)},\qquad \hat u_k(s)=\frac{\theta(s)}{\sigma(s)},$$

for $s\in[t_{k-1},t_k]$.

In summary, we state the following multi-person game.

Problem (G_Π). For $k=1,2,\cdots,N$, the state process of Player (k) is given by

$$\begin{cases}\mathrm{d}X_k(s)=[r(s)-c_k(s)+\theta(s)\sigma(s)u_k(s)]X_k(s)\mathrm{d}s+\sigma(s)u_k(s)X_k(s)\mathrm{d}W(s),\quad s\in[t_{k-1},t_k),\\ X_k(t_{k-1})=X_{k-1}(t_{k-1}),\end{cases}$$

with $X_0(t_0)=x$. He/she chooses control $(c_k(\cdot),u_k(\cdot))\in\mathcal{A}_k(t_{k-1},X_{k-1}(t_{k-1}))$ to maximize the performance functional

$$J^k_\Pi(x;(c_1(\cdot),u_1(\cdot)),(c_2(\cdot),u_2(\cdot)),\cdots,(c_N(\cdot),u_N(\cdot))):=J_k\left(t_{k-1},X_{k-1}(t_{k-1});c_k(\cdot),u_k(\cdot)\right).$$

Definition 8. For a given partition $\Pi\in\mathcal{P}[t,T]$, an N-tuple of controls $((\hat c_1(\cdot),\hat u_1(\cdot)),\cdots,(\hat c_N(\cdot),\hat u_N(\cdot)))\in\mathcal{A}_1(t_0,x)\times\cdots\times\mathcal{A}_N(t_{N-1},X_N(t_{N-1}))$ is called an open-loop Nash equilibrium of Problem (G_Π) if for all $k=1,2,\cdots,N$, and for any $(c_k(\cdot),u_k(\cdot))\in\mathcal{A}_k\left(t_{k-1},X_k(t_{k-1})\right)$,

$$\begin{aligned}&J^k_\Pi(x;(\hat c_1(\cdot),\hat u_1(\cdot)),\cdots,(\hat c_{k-1}(\cdot),\hat u_{k-1}(\cdot)),(c_k(\cdot),u_k(\cdot)),(\hat c_{k+1}(\cdot),\hat u_{k+1}(\cdot)),\cdots,(\hat c_N(\cdot),\hat u_N(\cdot)))\\ \le&J^k_\Pi(x;(\hat c_1(\cdot),\hat u_1(\cdot)),\cdots,(\hat c_{k-1}(\cdot),\hat u_{k-1}(\cdot)),(\hat c_k(\cdot),\hat u_k(\cdot)),(\hat c_{k+1}(\cdot),\hat u_{k+1}(\cdot)),\cdots,(\hat c_N(\cdot),\hat u_N(\cdot))).\end{aligned}$$

In this case, we call

$$(\hat c_\Pi(s),\hat u_\Pi(s)):=\sum_{k=1}^N(\hat c_k(s),\hat u_k(s))\mathbf{1}_{[t_{k-1},t_k)}(s)\in\mathcal{A}(t,x)$$

a Nash equilibrium control of Problem (G_Π) with initial state $(t,x)\in[0,T]\times(0,\infty)$.

We end this section by showing a Nash equilibrium control of Problem (G_Π). First of all, we introduce the following notation. For any partition $\Pi\in\mathcal{P}[t,T]$ and $s\in[t_0,t_N]$, let

$$l_\Pi(s)=\sum_{k=1}^{N} t_{k-1}\mathbf{1}_{[t_{k-1},t_k)}(s).$$

Let $\mathcal{D}[t,T]=\{(s,\tau)\,:\,t\le\tau\le s\le T\}$. For any $(s,\tau)\in\mathcal{D}[t,T]$, define the bounded function $g_\Pi(s,\tau)$ by

$$g_\Pi(s,\tau)=\sum_{k=1}^{N} g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(\tau).$$

For $t\le\tau\le s\le T$, let

$$\begin{aligned}
Y_\Pi(s,\tau) &= \sum_{k=1}^{N}\Big[Y_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s)+\mathcal{Y}_{k+1}(s)\mathbf{1}_{[t_k,t_N)}(s)\Big]\mathbf{1}_{[t_{k-1},t_k)}(\tau),\\
Z_\Pi(s,\tau) &= \sum_{k=1}^{N}\Big[Z_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s)+\mathcal{Z}_{k+1}(s)\mathbf{1}_{[t_k,t_N)}(s)\Big]\mathbf{1}_{[t_{k-1},t_k)}(\tau),
\end{aligned}$$

and

$$f_\Pi(s,\tau,y)=\sum_{k=1}^{N}\Big[f_k(s,y)\mathbf{1}_{[t_{k-1},t_k)}(s)+\mathcal{F}_{k+1}(s,y)\mathbf{1}_{[t_k,t_N)}(s)\Big]\mathbf{1}_{[t_{k-1},t_k)}(\tau).$$

It follows from Appendix B that for $(s,\tau)\in\mathcal{D}[t,T]$ we have

$$Y_\Pi(s,\tau)=\int_s^{t_N} f_\Pi(v,\tau,Y_\Pi(v,\tau))\mathrm{d}v+\int_s^{t_N} Z_\Pi(v,\tau)\mathrm{d}W(v). \tag{9}$$

From the construction of the multi-person game, we immediately get the following result.

Theorem 9. *For any partition $\Pi\in\mathcal{P}[t,T]$ and $\tau\in[t,T]$, let $g_\Pi(\cdot,\tau)$ be the unique solution of the ODE*

$$\begin{cases} g'_\Pi(s,\tau)=-g_\Pi(s,\tau)\frac{h'(s-l_\Pi(\tau))}{h(s-l_\Pi(\tau))}-1, \quad t\le\tau\le s\le t_N,\\ g_\Pi(t_N,\tau)=1,\end{cases}$$

where $g'_\Pi(s,\tau)$ is the partial derivative with respective to the first variable s. *For any $x\in(0,\infty)$, define $\hat X_\Pi(\cdot;t,x)$ by the SDE*

$$\begin{cases}\mathrm{d}\hat X_\Pi(s)=[r(s)-\hat c_\Pi(s)+\theta(s)\sigma(s)\hat u_\Pi(s)]\hat X_\Pi(s)\mathrm{d}s+\sigma(s)\hat u_\Pi(s)\hat X_\Pi(s)\mathrm{d}W(s), \quad s\in[t,t_N],\\ \hat X_\Pi(t)=x,\end{cases}$$

where

$$\hat c_\Pi(s)=\frac{1}{g_\Pi(s,s)}, \qquad \hat u_\Pi(s)=\frac{\theta(s)}{\sigma(s)}.$$

Then $\hat{X}_\Pi(\cdot;t,x)$ *and* $(\hat{c}_\Pi(\cdot),\hat{u}_\Pi(\cdot))$ *are equilibrium state process and Nash equilibrium control for Problem* (G_Π) *with initial state* $(t,x)\in[0,T]\times(0,\infty)$*, respectively. Furthermore, for any* $\tau\in[t,T]$*, let* $(Y_\Pi(\cdot,\tau),Z_\Pi(\cdot\tau))$ *be the solution of the BSDE*

$$\begin{cases} dY_\Pi(s,\tau)=-f_\Pi(s,\tau,Y_\Pi(s,\tau))ds-Z_\Pi(s,\tau)dW(s), & \tau\le s\le t_N,\\ Y_\Pi(t_N,\tau)=0 \end{cases}$$

which is parameterized by τ*. Then we have*

$$V_\Pi(t,x)=g_\Pi(t,t)\,[\ln x-Y_\Pi(t,t)]\,,$$

for any $(t,x)\in[0,T)\times(0,\infty)$.

4. Time-Consistent Equilibrium Strategies

Now, we are going to derive a time-consistent solution to the original Problem (N) by letting $\|\Pi\|\to 0$ (i.e., $N\to\infty$). The proofs of the results in this section are given in Appendix C. The following definition is similar to Definition 4.3 of Yong (2012a).

Definition 10. An adapted process $(\hat{c}(\cdot),\hat{u}(\cdot))\in\mathcal{A}(t,x)$ is called a time-consistent equilibrium strategy of Problem (N) with initial state $(t,x)\in[0,T]\times(0,\infty)$ if the following hold:

(i) The SDE

$$\begin{cases} d\hat{X}(s)=[r(s)-\hat{c}(s)+\theta(s)\sigma(s)\hat{u}(s)]\,\hat{X}(s)ds+\sigma(s)\hat{u}(s)\hat{X}(s)dW(s), & s\in[t,T],\\ \hat{X}(t)=x, \end{cases}$$

admits a unique solution $\hat{X}(\cdot)\equiv\hat{X}(\cdot;t,x,\hat{c}(\cdot),\hat{u}(\cdot))$.

(ii) Approximate optimality: For any $\varepsilon>0$, there exists a $\delta>0$ such that for any partition $\Pi\in\mathcal{P}[t,T]$ with $\|\Pi\|<\delta$, one has the following: For any $k=1,2,\cdots,N$, and $(c_k(\cdot),u_k(\cdot))\in\mathcal{A}_k(t_{k-1},\hat{X}(t_{k-1}))$,

$$J\Big(t_{k-1},\hat{X}(t_{k-1});(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[t_{k-1},t_N)}\Big)\ge J\Big(t_{k-1},\hat{X}(t_{k-1});(c_k(\cdot),u_k(\cdot))\oplus(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[t_k,t_N)}\Big)-\varepsilon, \quad (10)$$

where

$$\Big[(c_k(\cdot),u_k(\cdot))\oplus(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[t_k,t_N)}\Big](s)=\begin{cases}(c_k(s),u_k(s)), & s\in[t_{k-1},t_k),\\ (\hat{c}(s),\hat{u}(s)), & s\in[t_k,t_N).\end{cases}$$

For any $s\in[t,T)$, give a partition $\Pi\in\mathcal{P}[t,T]$ with $\|\Pi\|<\delta$ and $s=t_{k-1}$ for some $k=1,2,\cdots,N$. Then (10) implies that along the equilibrium state process $\hat{X}(\cdot)$, the time-consistent equilibrium strategy $(\hat{c}(\cdot),\hat{u}(\cdot))$ keeps the approximate optimality.

Definition 11. A function $V(\cdot,\cdot):[0,T]\times(0,\infty)\to\mathbb{R}$ is called a equilibrium value function for Problem (N) if

$$V(t,x) \quad = \quad J(t,x;(\hat{c}(\cdot),\hat{u}(\cdot)))\,.$$

For $(s,\tau)\in\mathcal{D}[t,T]$, let

$$\begin{aligned} g(s,\tau) &= \exp\left\{\int_s^T \frac{h'(v-\tau)}{h(v-\tau)}\mathrm{d}v\right\}+\int_s^T \exp\left\{\int_s^z \frac{h'(v-\tau)}{h(v-\tau)}\mathrm{d}v\right\}\mathrm{d}z \\ &= \frac{1}{h(s-\tau)}\left[h(T-\tau)+\int_s^T h(v-\tau)\mathrm{d}v\right], \end{aligned} \tag{11}$$

and $(Y(\cdot,\tau),Z(\cdot,\tau))$ be the solution of the BSDE

$$\begin{cases} \mathrm{d}Y(s,\tau)=-f(s,\tau,Y(s,\tau))\mathrm{d}s-Z(s,\tau)\mathrm{d}W(s), \quad \tau\le s\le T,\\ Y(T,\tau)=0, \end{cases} \tag{12}$$

where

$$f(s,\tau,y)=-\frac{1}{g(s,\tau)}y+\frac{1}{g(s,\tau)}\ln g(s,s)+\frac{1}{g(s,s)}-\frac{1}{2}\theta^2(s)-r(s).$$

Note that (12) is a flow of BSDEs parameterized by the variable τ (see, e.g. El Karoui et al. (1997, Section 2.4)). Furthermore, for $s\in[t,T]$, we define

$$\hat{c}(s)=\frac{1}{g(s,s)}, \qquad \hat{u}(s)=\frac{\theta(s)}{\sigma(s)}. \tag{13}$$

Now we have the main results of the paper.

Theorem 12. *We have*

$$\lim_{\|\Pi\|\to 0}\sup_{(s,\tau)\in\mathcal{D}[t,T]}|g_\Pi(s,\tau)-g(s,\tau)|=0,$$

and for any $\tau\in[t,T]$

$$\lim_{\|\Pi\|\to 0}\mathsf{E}\left[\sup_{s\in[\tau,T]}|Y_\Pi(s,\tau)-Y(s,\tau)|^2+\int_\tau^T|Z_\Pi(s,\tau)-Z(s,\tau)|^2\,\mathrm{d}s\right]=0.$$

Theorem 13. *The pair* $(\hat{c}(\cdot),\hat{u}(\cdot))$ *defined by (13) is a time-consistent equilibrium strategy of Problem (N) with initial state* $(t,x)\in[0,T]\times(0,\infty)$*, the corresponding equilibrium state process* $\hat{X}(\cdot)$ *and equilibrium value function are given by*

$$\begin{cases} \mathrm{d}\hat{X}(s)=\left[r(s)-\frac{1}{g(s,s)}+\theta^2(s)\right]\hat{X}(s)\mathrm{d}s+\theta(s)\hat{X}(s)\mathrm{d}W(s), \quad s\in[t,T],\\ \hat{X}(t)=x \end{cases} \tag{14}$$

and

$$V(t,x) = g(t,t)\,[\ln x - Y(t,t)]\,, \quad \forall (t,x) \in [0,T]\times(0,\infty), \tag{15}$$

respectively.

Remark 14. From the proof of Theorem 13 (see Appendix C), we know that the strategy given by (13) is admissible, and the equilibrium value function V is the limit of V_Π as $||\Pi||$ approaches to 0.

5. Concluding Remarks

We investigated a time-inconsistent consumption-investment problem with a general discount function and a logarithmic utility function in a non-Markovian framework. We followed Yong's approach to study an N-player differential game. Using the martingale method, we solved the optimization problem of each player and characterized their optimal strategies and value functions. We obtained a time-consistent equilibrium consumption-investment strategy and the corresponding equilibrium value function of the original problem.

The same problem is studied in Marín-Solano & Navas (2010) under a model with deterministic coefficients. The authors solve this problem with general utility functions for both naive and sophisticated agents. Specifically, the authors show that, for the logarithmic utility function, the naive strategy coincides with the one for the sophisticated agent. This special feature for the logarithmic utility function is also shown in Pollak (1968) and Marín-Solano & Navas (2009) which consider the cake-eating problem in deterministic models. Our results show that this coincidence is preserved for the model with stochastic coefficients. In fact, at any time $t \in [0,T]$ with wealth $x > 0$, a naive t-agent will solve Problem (C) in Appendix A.1 with $T_1 = t, T_2 = T, a = 1$ and $F = 0$. In this special case, the function $g(\cdot)$ given by (A.3) (to emphasize the dependence on the time t, here we denote it by $g_t(\cdot)$) becomes

$$g_t(s) = \exp\left\{\int_s^T \frac{h'(v-t)}{h(v-t)}\mathrm{d}v\right\} + \int_s^T \exp\left\{\int_s^z \frac{h'(v-t)}{h(v-t)}\mathrm{d}v\right\}\mathrm{d}z, \quad s \in [t,T].$$

Thus, from Theorem A.2, the naive strategy (denoted by $(\tilde{c}(\cdot),\tilde{u}(\cdot))$) is given by

$$\tilde{c}(t) = \frac{1}{g_t(t)}, \qquad \tilde{u}(t) = \frac{\theta(t)}{\sigma(t)}, \qquad t \in [0,T].$$

Note that $g_t(t)$ equals to $g(t,t)$ (with an abuse of notation g), where the function $g(\cdot,\cdot)$ is defined by (11). Thus, in our case, the naive strategy coincides with the time-consistent strategy given in Section 4. It also shows that the time-consistent strategy in our non-Markovian framework is essentially the same with the one obtained in Marín-Solano & Navas (2010) (note that $h(0) = 1$, in their paper the consumption strategy is the dollar amount and in our paper it is the proportion of the wealth).

Furthermore, similar to Marín-Solano & Navas (2010), the same problem can be considered with power and exponential utilities. However, there are some difficulties we want to remark here. For logarithmic utilities, it follows from Theorem 7 that the optimal strategy of Player (k), i.e. $(\hat{c}_k(\cdot),\hat{u}_k(\cdot))$ is independent of $(Y_k(\cdot),Z_k(\cdot))$. Consequently, $(\mathcal{Y}_k(\cdot),\mathcal{Z}_k(\cdot))$ which depends on $(\hat{c}_n(\cdot),\hat{u}_n(\cdot)),n=k,\cdots,N$, is also independent of $(Y_n(\cdot),Z_n(\cdot)),n=k,\cdots,N$. Thus, by defining $Y_\Pi(\cdot,\cdot)$,$Z_\Pi(\cdot,\cdot)$ and $f_\Pi(\cdot,\cdot,\cdot)$, we get a standard BSDE (with a parameter), i.e. Equation (9). Then letting the mesh size approaches to zero, we get the limit equation of (9), i.e. Equation (12) which is also a standard BSDE. Unfortunately, this is not the case for power and exponential utilities. From the results obtained by Cheridito & Hu (2011), it is easy to see that for power and exponential utilities the optimal strategy of Player (k) depends on $(Y_k(\cdot),Z_k(\cdot))$ which is the unique solution of some BSDE. Similarly, we can define $(\mathcal{Y}_k(\cdot),\mathcal{Z}_k(\cdot))$ as the unique solution to some BSDE. But in these cases, $(Y_n(\cdot),Z_n(\cdot)),n=k,\cdots,N$, appear in the driver of the BSDE satisfied by $(\mathcal{Y}_k(\cdot),\mathcal{Z}_k(\cdot))$. Similarly, we define $Y_\Pi(\cdot,\cdot),Z_\Pi(\cdot,\cdot)$ and $f_\Pi(\cdot,\cdot,\cdot,\cdot)$. Unlike Equations (9) and (12), we can not get any standard equations for $(Y_\Pi(\cdot,\cdot),Z_\Pi(\cdot,\cdot))$ and the limit process $(Y(\cdot,\cdot),Z(\cdot,\cdot))$. At the moment, it is not clear to us what kind of (non-standard) equation it will lead to and how we get the existence and uniqueness of the solution of the non-standard equation. Therefore, we leave the cases with power and exponential utilities in our future research.

Appendix A

A.1. A Time-Consistent Control Problem

In this appendix, we consider a time-consistent consumption-investment problem and give explicit solutions to the optimal strategy and value function in terms of the unique solution to a BSDE. The optimal control problems of all players in Section 3 are special cases of the problem studied in this section. The results of this section will be used frequently in the paper.

Given $0\le T_1\le T_2\le T$, for any $t\in[T_1,T_2)$ we consider the wealth process

$$\begin{cases}\mathrm{d}X^{c,u}(s)=[r(s)-c(s)+u(s)\sigma(s)\theta(s)]X^{c,u}(s)\mathrm{d}s+u(s)\sigma(s)X^{c,u}(s)\mathrm{d}W(s), & s\in[t,T_2],\\ X^{c,u}(t)=x,\end{cases}\tag{A.1}$$

where $x>0$. Note that the wealth process given by (A.1) is positive.

Assume that at time $s\in[T_1,T_2]$ the investor adopts the discount factor $h(s-T_1)$. Consider the performance functional

$$J_c(t,x;c(\cdot),u(\cdot))=\mathsf{E}_t\left[\int_t^{T_2}h(s-T_1)\ln\big(c(s)X^{c,u}(s)\big)\,\mathrm{d}s+h(T_2-T_1)a\big(\ln X^{c,u}(T_2)-F\big)\right],\tag{A.2}$$

where $a>0$ is a bounded constant and F is a bounded, $\mathcal{F}_{T_2}$-measurable random variable.

Definition A.1. For any $(t,x)\in[T_1,T_2)\times(0,\infty)$, an admissible strategy consists of a pair $(c(\cdot),u(\cdot))\in$

$C[t,T_2]\times\mathcal{U}[t,T_2]$ such that

$$\mathsf{E}_t\left[\int_t^{T_2}\left|\ln\left(c(s)X^{c,u}(s)\right)\right|\mathrm{d}s\right]<\infty.$$

We denote by $\mathcal{A}_c(t,x)$ the class of all such admissible strategies.

Problem (C)**.** For any $(t,x)\in[T_1,T_2)\times(0,\infty)$, find a strategy $(\hat{c}(\cdot),\hat{u}(\cdot))\in\mathcal{A}_c(t,x)$ such that

$$\begin{aligned} J_c(t,x;\hat{c}(\cdot),\hat{u}(\cdot)) &= V_c(t,x) \\ &:= \sup_{(c(\cdot),u(\cdot))\in\mathcal{A}_c(t,x)} J_c(t,x;c(\cdot),u(\cdot)). \end{aligned}$$

Since T_1 is fixed (i.e. it is viewed as a parameter), it is well-known that Problem (C) is a time-consistent control problem, i.e., if $(\hat{c}(\cdot),\hat{u}(\cdot))\in\mathcal{A}_c(t,x)$ is optimal for the initial pair (t,x), then

$$\begin{aligned} J_c(\tau,X^{\hat{c},\hat{u}}(\tau);(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[\tau,T_2]}) &= V_c(\tau,X^{\hat{c},\hat{u}}(\tau)) \\ &:= \sup_{(c(\cdot),u(\cdot))\in\mathcal{A}_c(\tau,X^{\hat{c},\hat{u}}(\tau))} J_c(\tau,X^{\hat{c},\hat{u}}(\tau);c(\cdot),u(\cdot)), \quad \forall\tau\in[t,T_2]. \end{aligned}$$

Define a bounded positive function $g(\cdot)$ on $[T_1,T_2]$ by the ODE

$$\begin{cases} g'(s)=-g(s)\frac{h'(s-T_1)}{h(s-T_1)}-1, & s\in[T_1,T_2], \\ g(T_2)=a. \end{cases}$$

It is easy to see

$$g(s)=a\exp\left\{\int_s^{T_2}\frac{h'(v-T_1)}{h(v-T_1)}\mathrm{d}v\right\}+\int_s^{T_2}\exp\left\{\int_s^{z}\frac{h'(v-T_1)}{h(v-T_1)}\mathrm{d}v\right\}\mathrm{d}z,\quad s\in[T_1,T_2]. \tag{A.3}$$

For $s\in[T_1,T_2]$, consider the BSDE

$$Y(s)=F+\int_s^{T_2}f(v,Y(v))\mathrm{d}v+\int_s^{T_2}Z(v)\mathrm{d}W(v) \tag{A.4}$$

with the driver

$$f(s,y)=-\frac{1}{2}\theta^2(s)-\frac{1}{g(s)}y+\frac{1}{g(s)}(\ln g(s)+1)-r(s).$$

Note that $f(s,y)$ satisfies linear grow in y and all the other terms are bounded, it is well know that (A.4) admits a unique solution $(Y,Z)\in\mathbb{H}^2_{T_1,T_2}(\mathbb{R})\times\mathbb{H}^2_{T_1,T_2}(\mathbb{R})$ (see, e.g. Pardoux & Peng (1990) and El Karoui et al. (1997)). Furthermore, it is easy to see that Y is bounded, i.e. $Y\in\mathbb{H}^\infty_{T_1,T_2}(\mathbb{R})$.

Theorem A.2. *For any* $(t,x)\in[T_1,T_2)\times(0,\infty)$*, the value function for Problem (C) is*

$$V_c(t,x)=h(t-T_1)g(t)\left[\ln x-Y(t)\right],$$

and the optimal consumption-investment strategy is given by

$$\hat{c}(s) = \frac{1}{g(s)}, \qquad \hat{u}(s) = \frac{\theta(s)}{\sigma(s)}, \tag{A.5}$$

for $s \in [t, T_2]$.

The proof of the above theorem is similar to Cheridito & Hu (2011, Section 4.1) and we omit the details here. The idea is to prove that the process

$$R^{c,u}(s) := h(s-T_1)g(s)\left[\ln X^{c,u}(s) - Y(s)\right] + \int_t^s h(v-T_1)\ln\left(c(v)X^{c,u}(v)\right)\mathrm{d}v,$$

is a supermartingale. Then $R^{c,u}(t) \geq \mathsf{E}_t\left[R^{c,u}(T_2)\right]$. Furthermore, it can be shown that the equality holds if and only if (A.5) is satisfied.

A.2. Player (N)

Let $g_N(\cdot)$ be a bounded positive function defined on $[t_{N-1}, t_N]$ by the ODE

$$\begin{cases} g_N'(s) = -g_N(s)\frac{h'(s-t_{N-1})}{h(s-t_{N-1})} - 1, \quad s \in [t_{N-1}, t_N], \\ g_N(t_N) = 1. \end{cases}$$

Note that Problem (C_N) is a special case of Problem (C). From Theorem A.2, we have the following result.

Theorem A.3. *For* $x_{N-1} \in (0, \infty)$*, the value function of Problem* (C_N) *is given by*

$$V_\Pi(t_{N-1}, x_{N-1}) = g_N(t_{N-1})\left(\ln x_{N-1} - Y_N(t_{N-1})\right),$$

where $Y_N(\cdot)$ *is defined by the unique solution* $(Y_N(\cdot), Z_N(\cdot)) \in \mathbb{H}^\infty_{t_{N-1},t_N}(\mathbb{R}) \times \mathbb{H}^2_{t_{N-1},t_N}(\mathbb{R})$ *of the BSDE*

$$Y_N(\tau) = \int_\tau^{t_N} f_N(s, Y_N(s))\mathrm{d}s + \int_\tau^{t_N} Z_N(s)\mathrm{d}W(s), \quad \tau \in [t_{N-1}, t_N],$$

with the driver

$$f_N(s,y) = -\frac{1}{g_N(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_N(s)}\left(\ln g_N(s) + 1\right) - r(s).$$

The optimal trading strategy $(\hat{c}_N(\cdot), \hat{u}_N(\cdot)) \in \mathcal{A}_N(t_{N-1}, x_{N-1})$ *is given by*

$$\hat{c}_N(s) = \frac{1}{g_N(s)}, \qquad \hat{u}_N(s) = \frac{\theta(s)}{\sigma(s)},$$

for $s \in [t_{N-1}, t_N]$*. The optimal wealth process* $\hat{X}_N(\cdot) \equiv \hat{X}_N(\cdot; t_{N-1}, x_{N-1})$ *is given by*

$$\begin{cases} \mathrm{d}\hat{X}_N(s) = \left[r(s) - \frac{1}{g_N(s)} + \theta^2(s)\right]\hat{X}_N(s)\mathrm{d}s + \theta(s)\hat{X}_N(s)\mathrm{d}W(s), \quad s \in [t_{N-1}, t_N], \\ \hat{X}_N(t_{N-1}) = x_{N-1}. \end{cases}$$

A.3. Player $(N-1)$

Proposition A.4. *For* $(\tau, x_\tau) \in [t_{N-1}, t_N] \times (0, \infty)$*, define*

$$\Theta_N(\tau, x_\tau) \quad := \quad \mathsf{E}_\tau\left[\int_\tau^{t_N} h(s - t_{N-2}) \ln\left(\hat{c}_N(s)\hat{X}_N(s; \tau, x_\tau)\right)\mathrm{d}s + h(t_N - t_{N-2}) \ln \hat{X}_N(t_N; \tau, x_\tau)\right].$$

We have

$$\Theta_N(\tau, x_\tau) = h(\tau - t_{N-2}) g_{N-1}(\tau)\left[\ln x_\tau - \mathcal{Y}_N(\tau)\right].$$

Proof. Similar to the proof of Proposition A.8. □

Similarly, Problem (C_{N-1}) is also a special case of Problem (C), from Theorem A.2 we have the following result.

Theorem A.5. *For any* $x_{N-2} \in [t_{N-2}, t_{N-1})$*, the value function of Problem* (C_{N-1}) *is given by*

$$V_\Pi(t_{N-2}, x_{N-2}) = g_{N-1}(t_{N-2})\left[\ln x_{N-2} - Y_{N-1}(t_{N-2})\right],$$

where $Y_{N-1}(\cdot)$ *is defined by the unique solution* $(Y_{N-1}(\cdot), Z_{N-1}(\cdot)) \in \mathbb{H}^\infty_{t_{N-2}, t_{N-1}}(\mathbb{R}) \times \mathbb{H}^2_{t_{N-2}, t_{N-1}}(\mathbb{R})$ *of the BSDE*

$$\begin{aligned} Y_{N-1}(\tau) \quad &= \quad \mathcal{Y}_N(t_{N-1}) + \int_\tau^{t_{N-1}} f_{N-1}(s, Y_{N-1}(s))\mathrm{d}s \\ &\quad + \int_\tau^{t_{N-1}} Z_{N-1}(s)\mathrm{d}W(s), \quad \tau \in [t_{N-2}, t_{N-1}], \end{aligned}$$

with the driver

$$f_{N-1}(s, y) = -\frac{1}{g_{N-1}(s)} y - \frac{1}{2}\theta^2(s) + \frac{1}{g_{N-1}(s)}\left(\ln g_{N-1}(s) + 1\right) - r(s).$$

The optimal trading strategy $(\hat{c}_{N-1}(\cdot), \hat{u}_{N-1}(\cdot)) \in \mathcal{A}_{N-1}(t_{N-2}, x_{N-2})$ *is given by*

$$\hat{c}_{N-1}(s) = \frac{1}{g_{N-1}(s)}, \qquad \hat{u}_{N-1}(s) = \frac{\theta(s)}{\sigma(s)},$$

for $s \in [t_{N-2}, t_{N-1}]$*. The optimal wealth process* $\hat{X}_{N-1}(\cdot) \equiv \hat{X}_{N-1}(\cdot; t_{N-2}, x_{N-2})$ *is given by*

$$\begin{cases} \mathrm{d}\hat{X}_{N-1}(s) = \left[r(s) - \frac{1}{g_{N-1}(s)} + \theta^2(s)\right]\hat{X}_{N-1}(s)\mathrm{d}s + \theta(s)\hat{X}_{N-1}(s)(s)\mathrm{d}W(s), \quad s \in [t_{N-2}, t_{N-1}), \\ \hat{X}_{N-1}(t_{N-2}) = x_{N-2}. \end{cases}$$

A.4. Player $(N-2)$

Proposition A.6. *For $(\tau, x_\tau) \in [t_{N-2}, t_N] \times (0, \infty)$, define*

$$\Theta_{N-1}(\tau, x_\tau) \;\; := \;\; \mathsf{E}_\tau\left[\int_\tau^{t_N} h(s-t_{N-3})\ln\Big(\bar{c}_{N-1}(s)\bar{X}_{N-1}(s;\tau,x_\tau)\Big)\mathrm{d}s + h(t_N-t_{N-3})\ln\bar{X}_{N-1}(t_N;\tau,x_\tau)\right].$$

We have

$$\Theta_{N-1}(\tau, x_\tau) = h(\tau-t_{N-3})g_{N-2}(\tau)\left[\ln x_\tau - \mathcal{Y}_{N-1}(\tau)\right].$$

Proof. Similar to the proof of Proposition A.8. □

The following theorem follows from Theorem A.2.

Theorem A.7. *For any $x_{N-3} \in (0, \infty)$, the value function of Problem (C_{N-2}) is given by*

$$V_\Pi(t_{N-3}, x_{N-3}) = g_{N-2}(t_{N-3})\left[\ln x_{N-3} - Y_{N-2}(t_{N-3})\right],$$

where $Y_{N-2}(\cdot)$ is defined by the unique solution $(Y_{N-2}(\cdot), Z_{N-2}(\cdot)) \in \mathbb{H}^\infty_{t_{N-3},t_{N-2}}(\mathbb{R}) \times \mathbb{H}^2_{t_{N-3},t_{N-2}}(\mathbb{R})$ of the BSDE

$$Y_{N-2}(\tau) = \mathcal{Y}_{N-1}(t_{N-2}) + \int_\tau^{t_{N-2}} f_{N-2}(s, Y_{N-2}(s))\mathrm{d}s + \int_\tau^{t_{N-2}} Z_{N-2}(s)\mathrm{d}W(s), \quad \tau \in [t_{N-3}, t_{N-2}],$$

with the driver

$$f_{N-2}(s,y) = -\frac{1}{2}\theta^2(s) - \frac{1}{g_{N-2}(s)}y + \frac{1}{g_{N-2}(s)}\left(\ln g_{N-2}(s) + 1\right) - r(s).$$

The optimal trading strategy $(\hat{c}_{N-2}(\cdot), \hat{u}_{N-2}(\cdot)) \in \mathcal{A}_{N-2}(t_{N-3}, x_{N-3})$ is given by

$$\hat{c}_{N-2}(s) = \frac{1}{g_{N-2}(s)}, \qquad \hat{u}_{N-2}(s) = \frac{\theta(s)}{\sigma(s)},$$

for $s \in [t_{N-3}, t_{N-2})$.

A.5. Player (k)

Proposition A.8. *For $(\tau, x_\tau) \in [t_k, t_N] \times (0, \infty)$, $k = 1, 2, \cdots, N-1$, define*

$$\Theta_{k+1}(\tau, x_\tau) \;\; := \;\; \mathsf{E}_\tau\left[\int_\tau^{t_N} h(s-t_{k-1})\ln\Big(\bar{c}_{k+1}(s)\bar{X}_{k+1}(s;\tau,x_\tau)\Big)\mathrm{d}s + h(t_N-t_{k-1})\ln\bar{X}_{k+1}(t_N;\tau,x_\tau)\right].$$

We have

$$\Theta_{k+1}(\tau, x_\tau) = h(\tau-t_{k-1})g_k(\tau)\left[\ln x_\tau - \mathcal{Y}_{k+1}(\tau)\right].$$

Proof. Define the process

$$R(s)=h(s-t_{k-1})g_k(s)\Big[\ln\bar{X}_{k+1}(s)-\mathcal{Y}_{k+1}(s)\Big]+\int_\tau^s h(v-t_{k-1})\ln\Big(\bar{c}_{k+1}(v)\bar{X}_{k+1}(v)\Big)\mathrm{d}v,$$

for $s\in[\tau,t_N]$. It is easy to check that

$$\begin{aligned}
R(\tau) &= h(\tau-t_{k-1})g_k(\tau)\Big[\ln\bar{X}_{k+1}(\tau)-\mathcal{Y}_{k+1}(\tau)\Big],\\
R(t_N) &= h(t_N-t_{k-1})\ln\bar{X}_{k+1}(t_N)+\int_\tau^{t_N} h(v-t_{k-1})\ln\Big(\bar{c}_{k+1}(v)\bar{X}_{k+1}(v)\Big)\mathrm{d}v,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}\Big(\ln\bar{X}_{k+1}(s)-\mathcal{Y}_{k+1}(s)\Big) &= \frac{1}{\bar{X}_{k+1}(s)}\mathrm{d}\bar{X}_{k+1}(s)-\frac{1}{2}\frac{1}{\bar{X}^2_{k+1}(s)}\Big[\sigma(s)\bar{u}_{k+1}(s)\bar{X}_{k+1}(s)\Big]^2\mathrm{d}s-\mathrm{d}\mathcal{Y}_{k+1}(s)\\
&= \Big[r(s)-\bar{c}_{k+1}(s)+\theta(s)\sigma(s)\bar{u}_{k+1}(s)-\frac{1}{2}\sigma^2(s)\bar{u}^2_{k+1}(s)+\mathcal{F}_{k+1}(s,\mathcal{Y}_{k+1}(s))\Big]\mathrm{d}s\\
&\quad+[\sigma(s)\bar{u}_{k+1}(s)+\mathcal{Z}_{k+1}(s)]\,\mathrm{d}W(s),
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}R(s) &= \Big[h'(s-t_{k-1})g_k(s)+h(s-t_{k-1})g_k'(s)\Big]\Big[\ln\bar{X}_{k+1}(s)-\mathcal{Y}_{k+1}(s)\Big]\mathrm{d}s\\
&\quad+h(s-t_{k-1})g_k(s)\mathrm{d}\Big(\ln\bar{X}_{k+1}(s)-\mathcal{Y}_{k+1}(s)\Big)+h(s-t_{k-1})\ln\Big(\bar{c}_{k+1}(s)\bar{X}_{k+1}(s)\Big)\mathrm{d}s\\
&= h(s-t_{k-1})g_k(s)\{[\sigma(s)\bar{u}_{k+1}(s)+\mathcal{Z}_{k+1}(s)]\,\mathrm{d}W(s)+A(s)\mathrm{d}s\},
\end{aligned}$$

where

$$\begin{aligned}
A(s) &= \left[\frac{h'(s-t_{k-1})}{h(s-t_{k-1})}+\frac{g_k'(s)}{g_k(s)}\right]\Big[\ln\bar{X}_{k+1}(s)-\mathcal{Y}_{k+1}(s)\Big]\\
&\quad+r(s)-\bar{c}_{k+1}(s)+\theta(s)\sigma(s)\bar{u}_{k+1}(s)-\frac{1}{2}\sigma^2(s)\bar{u}^2_{k+1}(s)\\
&\quad+\mathcal{F}_{k+1}(s,\mathcal{Y}_{k+1}(s))+\frac{1}{g_k(s)}\ln\Big(\bar{c}_{k+1}(s)\bar{X}_{k+1}(s)\Big)\\
&= \frac{1}{g_k(s)}\mathcal{Y}_{k+1}(s)+\theta(s)\sigma(s)\bar{u}_{k+1}(s)-\frac{1}{2}\sigma^2(s)\bar{u}^2_{k+1}(s)+\mathcal{F}_{k+1}(s,\mathcal{Y}_{k+1}(s))\\
&\quad+\frac{1}{g_k(s)}\ln\bar{c}_{k+1}(s)-\bar{u}_{k+1}(s)+r(s)\\
&= \frac{1}{g_k(s)}\mathcal{Y}_{k+1}(s)+\frac{1}{2}\theta^2(s)+\mathcal{F}_{k+1}(s,\mathcal{Y}_{k+1}(s))\\
&\quad-\frac{1}{g_k(s)}\sum_{n=k+1}^N\ln g_n(s)\mathbf{1}_{[t_{n-1},t_n)}(s)-\sum_{n=k+1}^N\frac{1}{g_n(s)}\mathbf{1}_{[t_{n-1},t_n)}(s)+r(s)\\
&= 0,
\end{aligned}$$

which implies that R is a local martingale. Since $\mathcal{Z}_{k+1}(\cdot)\in\mathbb{H}^2_{t_k,t_{k+1}}(\mathbb{R})$ and the other terms in the coefficient of $\mathrm{d}W(s)$ in $\mathrm{d}R(s)$ are bounded, we conclude that $R(s)$ is a martingale, i.e. $R(\tau)=\mathsf{E}_\tau[R(t_N)]$. □

Appendix B

Obviously, for any $\tau \in [t,T]$, the function $g_\Pi(\cdot,\tau)$ satisfies the ODE

$$\begin{cases} g'_\Pi(s,\tau) = -g_\Pi(s,\tau)\frac{h'(s-l_\Pi(\tau))}{h(s-l_\Pi(\tau))} - 1, \quad t \le \tau \le s \le t_N, \\ g_\Pi(t_N,\tau) = 1, \end{cases}$$

where $g'_\Pi(s,\tau)$ is the partial derivative with respective to the first variable s. It is easy to see that for $t \le \tau \le s \le T$,

$$\begin{aligned} g_\Pi(s,\tau) &= \sum_{k=1}^{N} g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\ &= \exp\left\{\int_s^{t_N} \frac{h'(v-l_\Pi(\tau))}{h(v-l_\Pi(\tau))}\mathrm{d}v\right\} + \int_s^{t_N} \exp\left\{\int_s^{z} \frac{h'(v-l_\Pi(\tau))}{h(v-l_\Pi(\tau))}\mathrm{d}v\right\}\mathrm{d}z \\ &= \frac{1}{h(s-l_\Pi(\tau))}\left[h(t_N - l_\Pi(\tau)) + \int_s^{t_N} h(z-l_\Pi(\tau))\mathrm{d}z\right] \end{aligned}$$

and

$$g_\Pi(s,s)\mathbf{1}_{[t_{k-1},t_k)}(s) = g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s). \tag{B.1}$$

Note that

$$\begin{aligned} \mathcal{F}_{k+1}(s,y) &= -\frac{1}{g_k(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_k(s)}\sum_{n=k+1}^{N}\left[\ln g_n(s)\right]\mathbf{1}_{[t_{n-1},t_n)}(s) \\ &\quad + \sum_{n=k+1}^{N}\left[\frac{1}{g_n(s)}\right]\mathbf{1}_{[t_{n-1},t_n)}(s) - r(s) \\ &= -\frac{1}{g_k(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_k(s)}\ln\left(\sum_{n=k+1}^{N} g_n(s)\mathbf{1}_{[t_{n-1},t_n)}(s)\right) \\ &\quad + \frac{1}{\sum_{n=k+1}^{N}\left(g_n(s)\mathbf{1}_{[t_{n-1},t_n)}(s)\right)} - r(s) \\ &= -\frac{1}{g_k(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_k(s)}\ln g_\Pi(s,s) + \frac{1}{g_\Pi(s,s)} - r(s), \quad t_{k-1} \le s < t_k. \end{aligned}$$

By (B.1), we have

$$\begin{aligned} f_\Pi(s,\tau,y) &= \sum_{k=1}^{N}\left[f_k(s,y)\mathbf{1}_{[t_{k-1},t_k)}(s) + \mathcal{F}_{k+1}(s,y)\mathbf{1}_{[t_k,t_N)}(s)\right]\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\ &= \left(\sum_{k=1}^{N} -\frac{1}{g_k(s)}\mathbf{1}_{[t_{k-1},t_k)}(\tau)\right)y - \frac{1}{2}\theta^2(s) - r(s) \\ &\quad + \sum_{k=1}^{N}\frac{1}{g_k(s)}\left[(\ln g_k(s))\mathbf{1}_{[t_{k-1},t_k)}(s) + (\ln g_\Pi(s,s))\mathbf{1}_{[t_k,t_N)}(s)\right]\mathbf{1}_{[t_{k-1},t_k)}(\tau) \end{aligned}$$

$$
\begin{aligned}
& + \sum_{k=1}^{N}\left(\frac{1}{g_k(s)}\mathbf{1}_{[t_{k-1},t_k)}(s) + \frac{1}{g_\Pi(s,s)}\mathbf{1}_{[t_k,t_N)}(s)\right)\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
= & -\frac{1}{\sum_{k=1}^{N} g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(\tau)}y - \frac{1}{2}\theta^2(s) - r(s) \\
& + \sum_{k=1}^{N}\frac{1}{g_k(s)}\ln\left(g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s) + g_\Pi(s,s)\mathbf{1}_{[t_k,t_N)}(s)\right)\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
& + \sum_{k=1}^{N}\frac{1}{g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s) + g_\Pi(s,s)\mathbf{1}_{[t_k,t_N)}(s)}\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
= & -\frac{1}{\sum_{k=1}^{N} g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(\tau)}y - \frac{1}{2}\theta^2(s) - r(s) \\
& + \sum_{k=1}^{N}\frac{1}{g_k(s)}\ln\left(g_\Pi(s,s)\mathbf{1}_{[t_{k-1},t_k)}(s) + g_\Pi(s,s)\mathbf{1}_{[t_k,t_N)}(s)\right)\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
& + \sum_{k=1}^{N}\frac{1}{g_\Pi(s,s)\mathbf{1}_{[t_{k-1},t_k)}(s) + g_\Pi(s,s)\mathbf{1}_{[t_k,t_N)}(s)}\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
= & -\frac{1}{g_\Pi(s,\tau)}y + \sum_{k=1}^{N}\frac{1}{g_k(s)}\ln g_\Pi(s,s)\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
& + \sum_{k=1}^{N}\frac{1}{g_\Pi(s,s)}\mathbf{1}_{[t_{k-1},t_k)}(\tau) - \frac{1}{2}\theta^2(s) - r(s) \\
= & -\frac{1}{g_\Pi(s,\tau)}y + \frac{1}{g_\Pi(s,\tau)}\ln g_\Pi(s,s) + \frac{1}{g_\Pi(s,s)} - \frac{1}{2}\theta^2(s) - r(s).
\end{aligned}
$$

Then, for $\tau \in [t_{k-1}, t_k)$, $s \in [\tau, t_k]$ and $k = 1, 2\cdots, N$, we have

$$
\begin{aligned}
Y_\Pi(s,\tau) &= \mathcal{Y}_{k+1}(t_k) + \int_s^{t_k} f_k(v, Y_\Pi(v,\tau))\mathrm{d}v + \int_s^{t_k} Z_\Pi(v,\tau)\mathrm{d}W(v) \\
&= \int_{t_k}^{t_N} \mathcal{F}_{k+1}(v, Y_\Pi(v,\tau))\mathrm{d}v + \int_{t_k}^{t_N} Z_\Pi(v,\tau)\mathrm{d}W(v) \\
&\quad + \int_s^{t_k} f_k(v, Y_\Pi(v,\tau))\mathrm{d}v + \int_s^{t_k} Z_\Pi(v,\tau)\mathrm{d}W(v) \\
&= \int_s^{t_N} f_\Pi(v,\tau, Y_\Pi(v,\tau))\mathrm{d}v + \int_s^{t_N} Z_\Pi(v,\tau)\mathrm{d}W(v).
\end{aligned}
$$

Obviously, the above equation holds for $\tau \in [t_{k-1}, t_k)$, $s \in [t_k, t_N]$ and $k = 1, 2\cdots, N$.

Appendix C

Proof of Theorem 12. Recalling Assumption 1, for $(s,\tau) \in \mathcal{D}[t,T]$, it is easy to see that

$$
|g_\Pi(s,\tau) - g(s,\tau)| \leq \frac{1}{h(s - l_\Pi(\tau))}\left(|h(T - l_\Pi(\tau)) - h(T-\tau)| + \int_s^T |h(v - l_\Pi(\tau)) - h(v-\tau)|\mathrm{d}v\right)
$$

$$
\begin{aligned}
&+\left|\frac{1}{h(s-l_\Pi(\tau))}-\frac{1}{h(s-\tau)}\right|\left[h(T-\tau)+\int_s^T h(v-\tau)\mathrm{d}v\right]\\
\leq\ & \frac{1}{h(T)}\left(1+\frac{1}{h(T)}\right)C\,(1+T)\,|l_\Pi(\tau)-\tau|\\
\leq\ & \frac{1}{h(T)}\left(1+\frac{1}{h(T)}\right)C\,(1+T)\,\|\Pi\|,
\end{aligned}
$$

which implies

$$\lim_{\|\Pi\|\to 0}\sup_{(s,\tau)\in\mathcal{D}[t,T]}|g_\Pi(s,\tau)-g(s,\tau)|=0.$$

Obviously, for any $(s,\tau)\in\mathcal{D}[t,T]$ we have

$$h(T)\leq g_\Pi(s,\tau),\quad g(s,\tau)\leq\frac{1}{h(T)}\,[1+T]\,.$$

Furthermore,

$$|\ln g_\Pi(s,\tau)-\ln g(s,\tau)|\leq\frac{1}{h(T)}\,|g_\Pi(s,\tau)-g(s,\tau)|\,.$$

Consequently, for fixed y,

$$
\begin{aligned}
&\int_\tau^T|f_\Pi(s,\tau,y)-f(s,\tau,y)|^2\,\mathrm{d}s\\
=&\int_\tau^T\left|-\frac{1}{g_\Pi(s,\tau)}y+\frac{1}{g_\Pi(s,\tau)}\ln g_\Pi(s,s)+\frac{1}{g_\Pi(s,s)}+\frac{1}{g(s,\tau)}y-\frac{1}{g(s,\tau)}\ln g(s,s)-\frac{1}{g(s,s)}\right|^2\mathrm{d}s\\
\leq&3\int_\tau^T\left[\left|\frac{1}{g_\Pi(s,\tau)}-\frac{1}{g(s,\tau)}\right|^2y^2+\left|\frac{1}{g_\Pi(s,\tau)}\ln g_\Pi(s,s)-\frac{1}{g(s,\tau)}\ln g(s,s)\right|^2+\left|\frac{1}{g_\Pi(s,s)}-\frac{1}{g(s,s)}\right|^2\right]\mathrm{d}s\\
=&3\int_\tau^T\left[\left|\frac{g(s,\tau)-g_\Pi(s,\tau)}{g_\Pi(s,\tau)g(s,\tau)}\right|^2y^2+\left|\frac{g(s,s)-g_\Pi(s,s)}{g_\Pi(s,s)g(s,s)}\right|^2\right]\mathrm{d}s\\
&+6\int_\tau^T\left[|\ln g_\Pi(s,s)-\ln g(s,s)|^2\,\frac{1}{g_\Pi^2(s,\tau)}+\left|\frac{g(s,\tau)-g_\Pi(s,\tau)}{g_\Pi(s,\tau)g(s,\tau)}\right|^2[\ln g(s,s)]^2\right]\mathrm{d}s\\
\leq&3\frac{1}{h^2(T)}\int_\tau^T\left[|g(s,\tau)-g_\Pi(s,\tau)|^2y^2+|g(s,s)-g_\Pi(s,s)|^2\right]\mathrm{d}s\\
&+6\frac{1}{h^2(T)}\int_\tau^T\left[|g(s,s)-g_\Pi(s,s)|^2\,\frac{1}{h^2(T)}+|g(s,\tau)-g_\Pi(s,\tau)|^2\,C'\right]\mathrm{d}s\\
\leq&C''\,\|\Pi\|^2,
\end{aligned}
$$

where

$$C'=\max\left\{[\ln h(T)]^2,\left[\ln\frac{1}{h(T)}\,[1+T]\right]^2\right\}$$

is a positive constant depending on T and y. Thus, for any fixed y

$$\lim_{\|\Pi\|\to 0}\sup_{\tau\in[t,T]}\int_\tau^T|f_\Pi(s,\tau,y)-f(s,\tau,y)|^2\,\mathrm{d}s=0.$$

It follows from the stability of BSDEs that for any $\tau \in [t,T]$

$$\lim_{\|\Pi\|\to 0} \mathsf{E}\left[\sup_{s\in[\tau,T]} |Y_\Pi(s,\tau)-Y(s,\tau)|^2 + \int_\tau^T |Z_\Pi(s,\tau)-Z(s,\tau)|^2\,\mathrm{d}s\right] = 0.$$

□

Lemma C.1. *Given any partition $\Pi \in \mathcal{P}[t,T]$ and any initial state $(t_{k-1},x) \in [0,T)\times(0,\infty)$, $k = 1,2,\cdots N$, let $\hat{X}(\cdot;t_{k-1},x)$ and $\hat{X}_\Pi(\cdot;t_{k-1},x)$ be the unique solutions to the SDEs*

$$\begin{cases}\mathrm{d}\hat{X}(s) = \left[r(s) - \frac{1}{g(s,s)} + \theta^2(s)\right]\hat{X}(s)\mathrm{d}s + \theta(s)\hat{X}(s)\mathrm{d}W(s), \quad s\in[t_{k-1},t_N],\\ \hat{X}(t_{k-1}) = x,\end{cases} \tag{C.1}$$

and

$$\begin{cases}\mathrm{d}\hat{X}_\Pi(s) = \left[r(s) - \frac{1}{g_\Pi(s,s)} + \theta^2(s)\right]\hat{X}_\Pi(s)\mathrm{d}s + \theta(s)\hat{X}_\Pi(s)\mathrm{d}W(s), \quad s\in[t_{k-1},t_N],\\ \hat{X}_\Pi(t_{k-1}) = x,\end{cases}$$

respectively. For $k = 1,2,\cdots N$ and $n \le k$, let

$$\begin{aligned}\Theta(n;k,x) &:= \mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\ln\left(\hat{c}(s)\hat{X}(s;t_{k-1},x)\right)\mathrm{d}s + h(t_N-t_{n-1})\ln\hat{X}(t_N;t_{k-1},x)\right],\\ \Theta_\Pi(n;k,x) &:= \mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\ln\left(\hat{c}_\Pi(s)\hat{X}_\Pi(s;t_{k-1},x)\right)\mathrm{d}s + h(t_N-t_{n-1})\ln\hat{X}_\Pi(t_N;t_{k-1},x)\right].\end{aligned}$$

Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|\Pi\| < \delta$, then

$$|\Theta(n;k,x) - \Theta_\Pi(n;k,x)| < \varepsilon, \quad \forall k = 1,2,\cdots N,\, n\le k,\, x\in(0,\infty).$$

Particularly, when $n = k$, we have

$$\left|J\left(t_{k-1},x;(\hat{c}_\Pi(\cdot),\hat{u}_\Pi(\cdot))\big|_{[t_{k-1},t_N)}\right) - J\left(t_{k-1},x;(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[t_{k-1},t_N)}\right)\right| < \varepsilon, \quad \forall k = 1,2,\cdots N,\, x\in(0,\infty).$$

Proof. It is easy to see that the unique solution to SDE (C.1) is given by

$$\hat{X}(s) = x\exp\left\{\int_{t_{k-1}}^s \left[r(v) - \frac{1}{g(v,v)} + \frac{1}{2}\theta^2(v)\right]\mathrm{d}v + \int_t^s \theta(v)\mathrm{d}W(v)\right\}, \quad s\in[t_{k-1},T].$$

For all $k = 1,2,\cdots,N$, and $n \le k$,

$$\begin{aligned}&\Theta(n;k,x)\\ =&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\ln\left(\hat{c}(s)\hat{X}(s)\right)\mathrm{d}s + h(t_N-t_{n-1})\ln\hat{X}(t_N)\right]\\ =&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\left(\ln x + \int_{t_{k-1}}^s\left[r(v) - \frac{1}{g(v,v)} + \frac{1}{2}\theta^2(v)\right]\mathrm{d}v + \int_{t_{k-1}}^s \theta(v)\mathrm{d}W(v) - \ln g(s,s)\right)\mathrm{d}s\right.\end{aligned}$$

$$
\begin{aligned}
&+h(t_N-t_{n-1})\left(\ln x+\int_{t_{k-1}}^{t_N}\left[r(s)-\frac{1}{g(s,s)}+\frac{1}{2}\theta^2(s)\right]\mathrm{d}s+\int_{t_{k-1}}^{t_N}\theta(s)\mathrm{d}W(s)\right)\Bigg]\\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\left[r(v)-\frac{1}{g(v,v)}+\frac{1}{2}\theta^2(v)\right]\mathrm{d}v-\ln g(s,s)\right)\mathrm{d}s\right.\\
&\left.+h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\left[r(s)-\frac{1}{g(s,s)}+\frac{1}{2}\theta^2(s)\right]\mathrm{d}s\right]+\left[\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\mathrm{d}s+h(t_N-t_{n-1})\right]\ln x \qquad \text{(C.2)}\\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\left[r(v)+\frac{1}{2}\theta^2(v)\right]\mathrm{d}v\right)\mathrm{d}s+h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\left[r(s)+\frac{1}{2}\theta^2(s)\right]\mathrm{d}s\right]\\
&-\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\frac{1}{g(v,v)}\mathrm{d}v+\ln g(s,s)\right)\mathrm{d}s-h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\frac{1}{g(s,s)}\mathrm{d}s\\
&+\left[\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\mathrm{d}s+h(t_N-t_{n-1})\right]\ln x.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Theta_\Pi(n;k,x)\\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\left[r(v)+\frac{1}{2}\theta^2(v)\right]\mathrm{d}v\right)\mathrm{d}s+h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\left[r(s)+\frac{1}{2}\theta^2(s)\right]\mathrm{d}s\right]\\
&-\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\frac{1}{g_\Pi(v,v)}\mathrm{d}v+\ln g_\Pi(s,s)\right)\mathrm{d}s-h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\frac{1}{g_\Pi(s,s)}\mathrm{d}s\\
&+\left[\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\mathrm{d}s+h(t_N-t_{n-1})\right]\ln x.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|\Theta(n;k,x)-\Theta_\Pi(n;k,x)|\\
=&\left|\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\frac{1}{g(v,v)}\mathrm{d}v+\ln g(s,s)\right)\mathrm{d}s+h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\frac{1}{g(s,s)}\mathrm{d}s\right.\\
&\left.-\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\frac{1}{g_\Pi(v,v)}\mathrm{d}v+\ln g_\Pi(s,s)\right)\mathrm{d}s-h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\frac{1}{g_\Pi(s,s)}\mathrm{d}s\right|.
\end{aligned}
$$

By Theorem 12 (and similar arguments of its proof), for any $\varepsilon>0$, there exists a $\delta>0$ such that for any partition $\Pi\in\mathcal{P}[0,T]$ with $\|\Pi\|<\delta$, it holds that

$$
|\Theta(n;k,x)-\Theta_\Pi(n;k,x)|<\varepsilon.
$$

□

Proof of Theorem 13. It is easy to see that the unique solution to SDE (14) is given by

$$
\hat{X}(s)=x\exp\left\{\int_t^s\left[r(v)-\frac{1}{g(v,v)}+\frac{1}{2}\theta^2(v)\right]\mathrm{d}v+\int_t^s\theta(v)\mathrm{d}W(v)\right\},\quad s\in[t,T].
$$

We are going to show the approximate optimality of $(\hat{c}(\cdot),\hat{u}(\cdot))$. From Lemma C.1, for any $\varepsilon>0$, there exists a $\delta > 0$ such that for any partition $\Pi \in \mathcal{P}[t,T]$ with $\|\Pi\| < \delta$, it holds that for all $k = 1,2,\cdots,N$,

$$J\Big(t_{k-1},\hat{X}(t_{k-1});(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[t_{k-1},t_N)}\Big) \geq J\Big(t_{k-1},\hat{X}(t_{k-1});(\hat{c}_\Pi(\cdot),\hat{u}_\Pi(\cdot))\big|_{[t_{k-1},t_N)}\Big) - \varepsilon.$$

Pick any $(c_k(\cdot),u_k(\cdot)) \in \mathcal{A}(t_{k-1},x)$, and let $X_k(\cdot) \equiv X_k(\cdot;t_{k-1},\hat{X}(t_{k-1}),c_k(\cdot),u_k(\cdot))$ be the solution to the SDE

$$\begin{cases} \mathrm{d}X_k(s) = [r(s) - c_k(s) + \theta(s)\sigma(s)u_k(s)]X_k(s)\mathrm{d}s + \sigma(s)u_k(s)X_k(s)\mathrm{d}W(s), & s \in [t_{k-1},t_k), \\ X_k(t_{k-1}) = \hat{X}(t_{k-1}). \end{cases}$$

By the construction of $(\hat{c}_\Pi(\cdot),\hat{u}_\Pi(\cdot))$, we have

$$J\Big(t_{k-1},\hat{X}(t_{k-1});(\hat{c}_\Pi(\cdot),\hat{u}_\Pi(\cdot))\big|_{[t_{k-1},t_N)}\Big) \geq J\Big(t_{k-1},\hat{X}(t_{k-1});\tilde{c}_\Pi(\cdot),\tilde{u}_\Pi(\cdot)\Big),$$

where

$$(\tilde{c}_\Pi(s),\tilde{u}_\Pi(s)) = \begin{cases} (c_k(s),u_k(s)), & s \in [t_{k-1},t_k), \\ (\hat{c}_\Pi(s),\hat{u}_\Pi(s)), & s \in [t_k,t_N]. \end{cases}$$

Note that

$$\begin{aligned}
&J\Big(t_{k-1},\hat{X}(t_{k-1});\tilde{c}_\Pi(\cdot),\tilde{u}_\Pi(\cdot)\Big) \\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} h(s-t_{k-1})\ln(c_k(s)X_k(s))\,\mathrm{d}s\right] \\
&+\mathsf{E}_{t_{k-1}}\left[\int_{t_k}^{t_N} h(s-t_{k-1})\ln\Big(\hat{c}_\Pi(s)\hat{X}_\Pi(s;t_k,X_k(t_k))\Big)\mathrm{d}s + h(t_N-t_{k-1})\ln\hat{X}_\Pi(t_N;t_k,X_k(t_k))\right] \\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} h(s-t_{k-1})\ln(c_k(s)X_k(s))\,\mathrm{d}s\right] + \mathsf{E}_{t_{k-1}}\left[\Theta_\Pi(k;k+1,X_k(t_k))\right] \\
\geq&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} h(s-t_{k-1})\ln(c_k(s)X_k(s))\,\mathrm{d}s\right] + \mathsf{E}_{t_{k-1}}\left[\Theta(k;k+1,X_k(t_k))\right] - \varepsilon \\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} h(s-t_{k-1})\ln(c_k(s)X_k(s))\,\mathrm{d}s\right] \\
&+\mathsf{E}_{t_{k-1}}\left[\int_{t_k}^{t_N} h(s-t_{k-1})\ln\Big(\hat{c}(s)\hat{X}(s;t_k,X_k(t_k))\Big)\mathrm{d}s + h(t_N-t_{k-1})\ln\hat{X}(t_N;t_k,X_k(t_k))\right] - \varepsilon \\
=&J\left(t_{k-1},x;(\tilde{c}(\cdot;t_{k-1},x),\tilde{u}(\cdot;t_{k-1},x))\right) - \varepsilon,
\end{aligned}$$

where the inequality follows from Lemma C.1. Thus we have

$$J\Big(t_{k-1},\hat{X}(t_{k-1});(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[t_{k-1},t_N)}\Big) \geq J\Big(t_{k-1},\hat{X}(t_{k-1});(\tilde{c}(\cdot),\tilde{u}(\cdot))\Big) - 2\varepsilon,$$

which implies the approximate optimality of $(\hat{c}(\cdot),\hat{u}(\cdot))$.

To obtain (15), note that $(Y(\cdot,\tau),Z(\cdot,\tau))$ is the unique solution to a linear BSDE for any $\tau \in [t,T]$.

Let $\rho(\cdot,\tau)$ be the solution to the ODE

$$\begin{cases} \mathrm{d}\rho(s,\tau) = -\frac{1}{g(s,\tau)}\rho(s,\tau)\mathrm{d}s, \quad s\in[\tau,T], \\ \mathrm{d}\rho(\tau,\tau) = 1, \end{cases}$$

then we have for $s\in[\tau,T]$,

$$\rho(s,\tau) = e^{-\int_\tau^s \frac{1}{g(v,\tau)}\mathrm{d}v},$$

and

$$Y(s,\tau) = \frac{1}{\rho(s,\tau)}\mathsf{E}_s\left[\int_s^T \rho(v,\tau)\left(\frac{1}{g(v,\tau)}\ln g(v,v) + \frac{1}{g(v,v)} - \frac{1}{2}\theta^2(v) - r(v)\right)\mathrm{d}v\right].$$

Noting that for any $\tau\in[t,T]$, the function $g(\cdot,\tau)$ satisfies the ODE

$$\begin{cases} g'(s,\tau) = -g(s,\tau)\frac{h'(s-\tau)}{h(s-\tau)} - 1, \quad 0\le\tau\le s\le t_N, \\ g(T,\tau) = 1, \end{cases}$$

it follows that

$$\begin{aligned} \rho(s,\tau) &= e^{\int_\tau^s\left[\frac{h'(v-\tau)}{h(v-\tau)}+\frac{g'(v,\tau)}{g(v,\tau)}\right]\mathrm{d}v} \\ &= \frac{h(s-\tau)g(s,\tau)}{g(\tau,\tau)}. \end{aligned}$$

Thus,

$$Y(s,\tau) = \frac{g(\tau,\tau)}{h(s-\tau)g(s,\tau)}\mathsf{E}_s\left[\int_s^T \frac{h(v-\tau)g(v,\tau)}{g(\tau,\tau)}\left(\frac{1}{g(v,\tau)}\ln g(v,v) + \frac{1}{g(v,v)} - \frac{1}{2}\theta^2(v) - r(v)\right)\mathrm{d}v\right],$$

and

$$\begin{aligned} g(t,t)Y(t,t) &= \mathsf{E}_t\left[\int_t^T h(s-t)g(s,t)\left(\frac{1}{g(s,t)}\ln g(s,s) + \frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\ &= \mathsf{E}_t\left[\int_t^T h(s-t)\ln g(s,s)\mathrm{d}s\right] \\ &\quad +\mathsf{E}_t\left[\int_t^T h(s-t)g(s,t)\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right]. \end{aligned}$$

Since

$$\begin{aligned} &\mathsf{E}_t\left[\int_t^T h(s-t)g(s,t)\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\ =&\mathsf{E}_t\left[\int_t^T h(s-t)\frac{1}{h(s-t)}\left[h(T-t) + \int_s^T h(v-t)\mathrm{d}v\right]\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\ =&\mathsf{E}_t\left[\int_t^T \left[h(T-t) + \int_s^T h(v-t)\mathrm{d}v\right]\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \end{aligned}$$

$$
\begin{aligned}
&= h(T-t)\mathsf{E}_t\left[\int_t^T \left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\
&\quad + \mathsf{E}_t\left[\int_t^T \int_s^T h(v-t)\mathrm{d}v\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\
&= h(T-t)\mathsf{E}_t\left[\int_t^T \left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\
&\quad + \mathsf{E}_t\left[\int_t^T h(v-t)\int_t^v \left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\mathrm{d}v\right],
\end{aligned}
$$

we have

$$
\begin{aligned}
g(t,t)Y(t,t) &= \mathsf{E}_t\left[\int_t^T h(s-t)\left(\int_t^s \left(\frac{1}{g(v,v)} - \frac{1}{2}\theta^2(v) - r(v)\right)\mathrm{d}v + \ln g(s,s)\right)\mathrm{d}s\right] \\
&\quad + h(T-t)\mathsf{E}_t\left[\int_t^T \left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right].
\end{aligned}
$$

Similar to (C.2)

$$
\begin{aligned}
J(t,x;(\hat{c}(\cdot),\hat{u}(\cdot))) &= g(t,t)\ln x + \mathsf{E}_t\left[\int_t^T h(s-t)\left(\int_t^s \left[r(v) - \frac{1}{g(v,v)} + \frac{1}{2}\theta^2(v)\right]\mathrm{d}v - \ln g(s,s)\right)\mathrm{d}s\right. \\
&\quad \left. + h(T-t)\int_t^T \left[r(s) - \frac{1}{\tilde{g}(s)} + \frac{1}{2}\theta^2(s)\right]\mathrm{d}s\right] \\
&= g(t,t)(\ln x - Y(t,t)).
\end{aligned}
$$

□

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