On a discrete-time risk model with claim correlated premiums

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Abstract

This paper proposes a discrete-time risk model that has a centain type of correlation between premiums and claim amounts. It is motivated by the well-known No Claims Discount system or bonus-malus system in the car insurance industry. Such a system penalises policyholders at fault in accidents by surcharges, and rewards claim-free years by discounts. For simplicity, only two levels of premium are considered in the given model and recursive formulae are derived for its ultimate ruin probabilities. Then the impact of the proposed correlation on ruin probabilities is examined through numerical examples. At last, the joint probability of ruin and deficit at ruin is also considered.

Keywords: Discrete-time risk model; No claims discount; Bonus-malus; Ruin probability; Deficit at ruin; Recursion

1 Introduction

The No Claims Discount (NCD) (or Bonus-Malus System) is a well-established system in general insurance industry worldwide, in particular for car insurances. For instance, in Australia, car insurance policyholders have the chance to renew their policies at a discounted level of premium if they made no claim at-fault in previous policy year. In another word, safe drivers are awarded by paying less premiums and "bad" drivers are penalised by paying more. In practice, the rules are fairly complicated and vary among insurers, but in general this arrangement encourages drivers to drive more safely and also leads to a reduction of the number of small claims. The Bonus-Malus has been studied extensively by many researchers in the past, see Denuit et al. (2007), Frangos and Vrontos (2001), Lemaire (1995) and Tremblay (1992) and the references therein.

Motivated by the type of correlation between premiums and claims embedded in the NCD system, we consider a discrete-time risk model with two premium levels that are dependent on claim amounts. Dufresne (1988) considered the stationary distributions of a bonus-malus system and showed that it can be computed recursively. It is further shown that there is an intrinsic relationship between such a stationary distribution and the probability of ruin in the risk theoretical model.

Let $\mathbb{N}^+ = \{1, 2, \ldots\}$. Denote U_n the amount of surplus of the insurer at time n, $n \in \mathbb{N}$, which has the form

$$U_n = u + \sum_{i=1}^{n} C_i - \sum_{i=1}^{n} X_i, \tag{1}$$

where $u \in \mathbb{N}$ is a constant initial surplus and $\{X_i\}_{i\in\mathbb{N}^+}$ form an independent and identically distributed (i.i.d.) random variables sequence with X_i denoting the total claim amount in period $i \in \mathbb{N}^+$. And C_i is the amount of premiums the insurer receives at the beginning of period i satisfying the following conditions, for $n \in \mathbb{N}^+$:

$$\begin{aligned} &\Pr(C_{n+1} = c | X_n = 0) = 0, \qquad \Pr(C_{n+1} = \theta c | X_n = 0) = 1, \\ &\Pr(C_{n+1} = c | X_n > 0) = 1, \qquad \Pr(C_{n+1} = \theta c | X_n > 0) = 0, \end{aligned}$$

where θ is a constant in $(0, 1]$ and c is a constant denoting the full premium level. Then θc means the discounted premium level. These conditions mean that if in period n there was no claim made, then the insurance company would charge lower premiums to its policyholders in next time period. Otherwise, the policyholders need to pay full premiums at renewal. In the present paper, we shall further assume that X_i only takes values 0 or 1 with probabilities $q = 1 - p$ and $p\,(0 < p < 1)$ respectively.

For a new insurance portfolio, it makes sense to charge full premiums for the first time period, i.e. $C_1 = c$, so we define

$$T = \min\left\{ n \in \mathbb{N}^+ : u + c + \sum_{i=2}^{n} C_i - \sum_{i=1}^{n} X_i < 0 \right\}$$

to be the time of ruin with $\sum_{i=2}^{1} = 0$. And $\psi(u) = \Pr(T < \infty)$ is the ultimate ruin probability for the discrete-time surplus process given in (1). A trivial observation about the ruin probabilities is $\psi(u) = 1$, for any $u < 0$. A positive safety loading condition for the model is $p < \frac{\theta c}{1-(1-\theta)c}$, which is calculated using the given assumptions for premiums.

If ruin occurs, then $|U_T| = y$ denotes the deficit at ruin satisfying $0 < y \leq 1$. We define

$$\varphi(u, y) = \begin{cases} \Pr(T < \infty, |U_T| = y | U(0) = u) & u \geq 0, \\ \delta_{-u,y} & u < 0, \end{cases}$$

which describes the probability that ruin occurs and the deficit at ruin equals y where $\delta_{-u,y}$ is an indicator function of $\{-u = y\}$. Obviously, $\delta_{-u,y} = \delta_{u,-y}$. Further, $\varphi(u, y)/\psi(u)$ denotes the probability function of the deficit at ruin given that ruin has occured.

The rest of the paper is organised as follows. Section 2 considers the above defined ultimate ruin probability $\psi(u)$ and derives recursive formulae for computation purposes. Explicit expressions are obtained for certain ranges of initial surplus. Section 3 works on the initial values for $\psi(u)$ to enable the usage of the obtained recursions. Section 4 studies the impact of the proposed correlation between premiums and claims on the ruin probabilities and numerical examples are provided to illustrate the impact. At last, in section 5, the deficit at ruin is considered using similar techniques to section 2.

2 Recursive formulae for $\psi(u)$

In this section, we shall derive recursive formulae for ruin probabilities $\psi(u)$ with the method used in Wagner (2001). Wagner (2001) considered a two-state Markov risk model where the state of a homogeneous Markov chain at any given time determines the corresponding claim amount. Since the risk model considered in this paper also has two states, with respect to premiums, the method proposed in Wagner (2001) can be used here.

Considering the first time period, we obtain the following recursion:

$$\psi(u) = q\psi\left(u + \theta c\right) + p\psi(u + c - 1). \tag{2}$$

The right-hand side of (2) includes two cases of claim occurrence in period 1. If there is no claim in the first time period, then $U_1 = u + c$, and $C_2 = \theta c$. If we deduct an amount of $(1-\theta)c$ from U_1 and combine it with C_2 then it is equivalent to a surplus process renewed at time 1 with an initial surplus $u + \theta c$. Otherwise, if there is a positive claim of 1 in period 1, then the surplus process is renewed at time 1 with an initial surplus $u + c - 1$.

Next we shall examine two ranges of u. For $0 \leq u < 1 - c$, equation (2) becomes

$$\psi(u) - q\psi\left(u + \theta c\right) = p. \tag{3}$$

And for $u \geq 1 - c$, we have

$$\psi(u) - q\psi\left(u + \theta c\right) = p\psi(u + c - 1). \tag{4}$$

Combining (3)-(4) we have

$$\psi(u) = \begin{cases} \frac{1}{q}[\psi(u - \theta c) - p] & \theta c \leq u < 1 - (1-\theta)c, \\ \frac{1}{q}[\psi(u - \theta c) - p\psi(u + (1-\theta)c - 1)] & u \geq 1 - (1-\theta)c. \end{cases} \tag{5}$$

To deal with the non-integers in the above recursion, we assume that $c = \frac{K_1}{N}$ and $\theta c = \frac{K_2}{N}$, where K_1, K_2 and N are all positive integers with $N >> K_1 \geq K_2$, $N > 1$ and their highest common factor is 1. Then the safety loading condition becomes $p < \frac{K_2}{N + K_2 - K_1}$. We further assume that u is a multiple of $\frac{1}{N}$. Writing $\xi(k) = \psi(\frac{k}{N})$, we obtain

$$\xi(k) = \begin{cases} \frac{1}{q}[\xi(k - K_2) - p] & K_2 \leq k < K_2 + N - K_1, \\ \frac{1}{q}[\xi(k - K_2) - p\xi(k - K_2 - N + K_1)] & k \geq K_2 + N - K_1. \end{cases} \tag{6}$$

Given $\xi(i)$, $i = 0, 1, \ldots, K_2 - 1$, using (6) one can obtain, for $K_2 \leq k \leq K_2 + N - K_1 - 1$, the following explicit expression of $\xi(k)$

$$\xi(k) = \frac{\xi(i) + q^j - 1}{q^j} = 1 - \frac{1}{q^j}\left[1 - \xi(i)\right], \tag{7}$$

where integers i and j are the remainder and quotient from dividing k by K_2, i.e. $k = i + jK_2$, $0 \leq i \leq K_2 - 1, j \geq 1$. For $k \geq K_2 + N - K_1$, $\xi(k)$ can be calculated recursively using the second part of formula (6).

3 The initial values of $\psi(u)$

Having obtained the explicit expression together with a recursive formula for $\xi(k)$, next we need to determine the initial values $\xi(i)$, $i = 0, 1, \ldots, K_2 - 1$. Without knowing them, one will not be able to apply the obtained results. The following derivation again follows Wagner (2001).

Suppose that $nK_2 > K_2 + N - K_1$. Let J_1 and I_1 be the quotient and remainder from dividing $K_2 + N - K_1 - 1$ by K_2. Due to the number of initial values to determine and the complexity embedded in the general setup of N, K_1, K_2, we need to introduce one more restriction before we go on that is $N - K_1$ is a multiple of K_2, i.e. $N - K_1 = J_1 K_2$. It implies that $I_1 = K_2 - 1$. Then we have

$$\begin{aligned}
\xi(nK_2) - \xi(0) &= \sum_{j=1}^{n} \Big[\xi(jK_2) - \xi((j-1)K_2)\Big] \\
&= \sum_{j=1}^{J_1} \Big[p\xi(jK_2) - p\Big] + \sum_{j=J_1+1}^{n} \Big[p\xi(jK_2) - p\xi((j-1)K_2 + K_1 - N)\Big] \\
&= p\sum_{j=1}^{n} \xi(jK_2) - p\sum_{j=0}^{n-J_1-1} \xi(jK_2) - pJ_1 \\
&= p\sum_{j=n-J_1}^{n} \xi(jK_2) - p\xi(0) - pJ_1.
\end{aligned}$$

When $u \to \infty$, $\xi(u)$ tends to zero, so letting $n \to \infty$ the above equation becomes

$$-\xi(0) = -p\xi(0) - pJ_1,$$

which gives $\xi(0) = \frac{pJ_1}{1-p}$. It is less than 1 according to the safety loading condition.

Similarly, for $i = 1, 2, \ldots, K_2 - 1$, we know $J_1 K_2 + i \le K_2 + N - K_1 - 1$ also holds. Therefore,

$$\begin{aligned}
\xi(nK_2 + i) - \xi(i) &= \sum_{j=1}^{n} \Big[\xi(jK_2 + i) - \xi((j-1)K_2 + i)\Big] \\
&= \sum_{j=1}^{J_1} \Big[p\xi(jK_2 + i) - p\Big] + \sum_{j=J_1+1}^{n} \Big[p\xi(jK_2 + i) - p\xi((j-1)K_2 + i + K_1 - N)\Big] \\
&= p\sum_{j=n-J_1}^{n} \xi(jK_2 + i) - p\xi(i) - pJ_1.
\end{aligned}$$

Letting $n \to \infty$ in the above equation and solving for $\xi(i)$ gives

$$\xi(i) = \frac{pJ_1}{1-p},$$

which is equal to $\xi(0)$. As a result, one can see that, for $j \in \mathbb{N}$,

$$\xi(jK_2) = \xi(1 + jK_2) = \cdots = \xi(K_2 - 1 + jK_2).$$

Since $\xi(k)$ is equivalent to $\psi(\frac{k}{N})$, we have obtained a complete formula for calculating the ruin probability $\psi(u)$.

Remarks.

- To be able to determine the initial values, $\xi(i)$, $i = 0, 1, \ldots, K_2 - 1$, an assumption was made at the beginning of this section, i.e. $N - K_1$ is a multiple of K_2. This assumption introduces a restriction on choosing values of N, K_1 and K_2. Since normally N is much bigger than K_1 and K_2, we do have a certain level of freedom to choose appropriate N, K_1 and K_2 values according to the given real problem. In many cases we may have to use numbers that are as close as possible to our expectation. It will certainly cause some inaccuracy regarding final results, but we believe the model and formulae given in this paper still can provide us some useful information on how the discounted premiums are going to affect the overall ruin probabilities.

- One simple example of how to choose appropriate N, K_1 and K_2 values is:

  - if we know the maximum claim size is around 100 times the full annual premium and the insurer is considering a 10% discount ($\theta = 0.9$), then $N = 1000, K_1 = 10$ and $K_2 = 9$ and the condition is satisfied.
  - for a 15% ($\theta = 0.85$) discount, we could use $N = 2009, K_1 = 20$ and $K_2 = 17$, and the claim size is approximately 100 (100.45) times the full premium level.

  For discrete risk models with non-integer irregular premiums/ surpluses, how to build a usable recursive framework for calculation purposes is still an open problem. Although the attempt in this paper does not cover the most general case, it might give readers a hint when searching for a usable solution.

- A very special case is that $K_2 = 1$, which means the discounted premium level θc is $\frac{1}{N}$ and the full premium is a multiple of the discounted one. From the calculation point of view, it can simplify all the previous results significantly. On the other hand, practically, it only covers few cases of discount, eg $\theta = 1, 0.5, 0.25, 0.125, \ldots$. In real life, a general insurer may be happy to give policyholders a 20% discount on premiums under certain conditions, but a discount over 60% is not often seen. Therefore, we are not going to put much attention on this case in the sequel.

4 The impact of variable premiums on $\psi(u)$

Based on Section 2 & 3 we develop the following algorithm for calculating $\xi(k), k \in \mathbb{N}$:

(i) Calculate J_1, j_k and i_k, where $J_1 = \frac{N-K_1}{K_2}$, and the latter two are the quotient and remainder from dividing k by K_2.

(ii) Calculate $\xi(k)$ as follows:

$$\xi(k) = \begin{cases} 1 - \frac{q-J_1p}{q^{j_k+1}} & 0 \leq j_k \leq J_1, \\ \frac{1}{q}\xi(k-K_2) - \frac{p}{q}\xi(k-(J_1+1)K_2) & j_k \geq J_1+1. \end{cases}$$

(iii) In step (ii), the second case needs to be calculated recursively.

Using the above algorithm, the ruin probability $\psi(\frac{k}{N})$ can be calculated for any given $k \in \mathbb{N}$. Also, if results are required for a set of k values, then the results for smaller k's should be stored for the calculation involving bigger k values.

Remarks. From step (ii), one can see that, for any $i = 0, 1, \ldots, K_2-1$ and $j \in \mathbb{N}$,

$$\xi(i + jK_2) = \xi(jK_2)$$

and

$$\xi(jK_2) = \begin{cases} 1 - \frac{q-J_1p}{q^{j+1}} & 0 \leq j \leq J_1, \\ \frac{1}{q}\xi((j-1)K_2) - \frac{p}{q}\xi((j-J_1-1)K_2) & j \geq J_1+1. \end{cases}$$

In the following, we shall examine the impact of the premium discounts on the ultimate ruin probability $\psi(u)$ through two numerical examples. We will illustrate how much the ruin probability will change with different level of discount on premiums under the assumptions of our model.

Example 1. In this example, we consider the following five cases:

(1) $N = 4,000, K_1 = 40, K_2 = 33, p = 0.008$;

(2) $N = 2,009, K_1 = 20, K_2 = 17, p = 0.008$;

(3) $N = 1,000, K_1 = 10, K_2 = 9, p = 0.008$;

(4) $N = 1,996, K_1 = 20, K_2 = 19, p = 0.008$;

(5) $N = 100, K_1 = 1, K_2 = 1, p = 0.008$.

They share the same claim size distribution, the same full premium level (c=0.01 where cases (2) and (4) are approximately equal) but different expected premium per time period. Some quantities of interest regarding these five cases are summarised in Table 1. Note here case (5) is a simplified situation which leads to a compound binomial risk model.

Case	c	θ	$E[X_1]$	$E[C_n]$	$\frac{E[C_n]}{E[X_1]} - 1$	J_1
(1)	0.01	0.825	0.008	0.008264	3.30%	120
(2)	0.01	0.85	0.008	0.008474	5.93%	117
(3)	0.01	0.90	0.008	0.009008	12.60%	110
(4)	0.01	0.95	0.008	0.009523	19.04%	104
(5)	0.01	1	0.008	0.01	25.00%	99

Table 1: Some characteristics of Example 1.

An obvious difference among the above five cases is the safety loading which is in the column entitled $\frac{E[C_n]}{E[X_1]} - 1$. Clearly the safety loading increases from case (1) to (5) significantly.

In Table 2 we provide ruin probability $\psi(u)$ values for selected u values in each case given above, where the superscript (i) corresponds to case (i). Bear in mind that although most of the u values are not big, the number of steps taken to obtain the ruin probabilities could be large. For example, to calculate $\psi^{(1)}(10)$, a little more than 1,200 steps of calculations are required.

One can see from Tables 1 & 2 that

- under the given conditions of Example 1, although the full premiums and claim size distributions are the same over the five cases, the ultimate ruin probabilities $\psi(u)$ are very different among them;
- case (5) has the lowest ruin prob.'s & the lowest (zero) discount, and case (1) has the highest ruin prob.'s & the highest discount level as well;

u	$\psi^{(1)}(u)$	$\psi^{(2)}(u)$	$\psi^{(3)}(u)$	$\psi^{(4)}(u)$	$\psi^{(5)}(u)$
0.0	0.9677	0.9435	0.8871	0.8387	0.7984
0.1	0.9645	0.9383	0.8767	0.8252	0.7815
0.2	0.9609	0.9321	0.8653	0.8091	0.7633
0.3	0.9569	0.9252	0.8528	0.7931	0.7435
0.4	0.9526	0.9177	0.8392	0.7740	0.7220
0.5	0.9478	0.9101	0.8244	0.7551	0.6987
0.6	0.9425	0.9009	0.8082	0.7325	0.6735
0.7	0.9367	0.8909	0.7904	0.7101	0.6462
0.8	0.9303	0.8799	0.7711	0.6833	0.6167
0.9	0.9232	0.8677	0.7499	0.6568	0.5846
1.0	0.9150	0.8548	0.7255	0.6264	0.5515
1.5	0.8876	0.8099	0.6510	0.5355	0.4513
2.0	0.8586	0.7640	0.5771	0.4492	0.3616
2.5	0.8313	0.7215	0.5140	0.3795	0.2913
3.0	0.8044	0.6811	0.4565	0.3193	0.2344
3.5	0.7784	0.6430	0.4063	0.2695	0.1885
4.0	0.7536	0.6070	0.3608	0.2267	0.1517
4.5	0.7293	0.5731	0.3211	0.1914	0.1221
5.0	0.7060	0.5410	0.2852	0.1610	0.0982
6.0	0.6611	0.4822	0.2255	0.1144	0.0636
7.0	0.6194	0.4293	0.1782	0.0812	0.0412
8.0	0.5802	0.3826	0.1409	0.0577	0.0266
9.0	0.5436	0.3410	0.1114	0.0410	0.0172
10.0	0.5093	0.3039	0.0879	0.0291	0.0112
20.0	0.2648	0.0959	0.0084	0.0010	0.0001

Table 2: Some values of $\psi(u)$ of Example 1.

- the above point is consistent with the trend of safety loadings of the five cases;
- as u increases, the differences of the ruin prob.'s between the five cases increase quickly, much higher than the differences between the discount levels.

Example 2. In this example, we reconsider the above five cases in Example 1 with changed probabilities for claims:

(1) $N = 4,000, K_1 = 40, K_2 = 33, p = 0.0075$;

(2) $N = 2,009, K_1 = 20, K_2 = 17, p = 0.0077$;

(3) $N = 1,000, K_1 = 10, K_2 = 9, p = 0.0082$;

(4) $N = 1,996, K_1 = 20, K_2 = 19, p = 0.0087$;

(5) $N = 100, K_1 = 1, K_2 = 1, p = 0.0091$.

The full premiums and discount levels remain the same but claim size distributions vary across the five cases such that their safty loadings stay roughly at the same level (10%). Table 3 summarises some quantities of interest regarding these five cases.

Case	c	θ	$E[X_1]$	$E[C_n]$	$\frac{E[C_n]}{E[X_1]} - 1$	J_0
(1)	0.01	0.825	0.0075	0.008264	10.19%	120
(2)	0.01	0.85	0.0077	0.008474	10.05%	117
(3)	0.01	0.90	0.0082	0.009008	9.85%	110
(4)	0.01	0.95	0.0087	0.009523	9.46%	104
(5)	0.01	1.00	0.0091	0.01	9.89%	99

Table 3: Some characteristics of Example 2.

Similar to Table 2, Table 4 summarises $\psi(u)$ values of the above five cases for selected u values. Comparing Tables 2 and 4 one can see the patterns shown in each table are totally different. In Table 4, the ruin probabilities of case (1) – (5) do not differ that much for any given u values. Instead, they are very close to each other displaying an increasing trend from case (1), (2), (5), (3) to (4). It implies that under the assumptions given in Example 2, ruin probabilities do not have the same quantitative orders as the premium discount levels. However, the trend among ruin probabilities in the five cases again is consistent with their safety loadings, i.e. ruin probability increases as safety loading decreases.

u	$\psi^{(1)}(u)$	$\psi^{(2)}(u)$	$\psi^{(3)}(u)$	$\psi^{(4)}(u)$	$\psi^{(5)}(u)$
0.0	0.9068	0.9079	0.9095	0.9127	0.9092
0.1	0.8980	0.8997	0.9009	0.9048	0.9005
0.2	0.8883	0.8900	0.8915	0.8952	0.8910
0.3	0.8778	0.8793	0.8812	0.8856	0.8805
0.4	0.8662	0.8675	0.8699	0.8741	0.8691
0.5	0.8536	0.8547	0.8576	0.8625	0.8565
0.6	0.8397	0.8418	0.8441	0.8487	0.8428
0.7	0.8246	0.8264	0.8293	0.8349	0.8278
0.8	0.8080	0.8095	0.8131	0.8182	0.8113
0.9	0.7883	0.7910	0.7937	0.8016	0.7932
1.0	0.7690	0.7714	0.7749	0.7824	0.7742
1.5	0.7034	0.7058	0.7108	0.7199	0.7093
2.0	0.6374	0.6408	0.6459	0.6565	0.6449
2.5	0.5787	0.5830	0.5890	0.6008	0.5874
3.0	0.5259	0.5300	0.5360	0.5486	0.5348
3.5	0.4772	0.4819	0.4885	0.5018	0.4869
4.0	0.4338	0.4382	0.4446	0.4582	0.4434
4.5	0.3936	0.3984	0.4045	0.4192	0.4037
5.0	0.3572	0.3622	0.3688	0.3828	0.3675
6.0	0.2946	0.2990	0.3059	0.3197	0.3047
7.0	0.2430	0.2472	0.2537	0.2671	0.2526
8.0	0.2004	0.2043	0.2105	0.2231	0.2094
9.0	0.1653	0.1689	0.1743	0.1863	0.1736
10.0	0.1361	0.1397	0.1446	0.1557	0.1439
20.0	0.0198	0.0208	0.0223	0.0257	0.0220

Table 4: Some values of $\psi(u)$ of Example 2.

5 The deficit at ruin

In this section, we shall consider one more function of interest, $\varphi(u, y)$, the probability that ruin occurs and the deficit at ruin equals $y, y > 0$, for the risk model (1). Note that Wagner (2002) also considered this probability for the Markov risk model defined in Wagner (2001). Under the same assumptions with respect to u, c and θc, i.e. $c = \frac{K_1}{N}$, $\theta c = \frac{K_2}{N}$, u is a multiple of $\frac{1}{N}$ and $N - K_1 = J_1 K_2$, we know that any possible deficit y is also a multiple of $\frac{1}{N}$ and $y = \frac{z}{N}$, $z = 1, 2, \ldots, N - K_2$. Clearly, for $u \geq 0$,

$$\psi(u) = \sum_{z=1}^{N-K_2} \varphi(u, \frac{z}{N}).$$

Given $y > 0$, considering the first time period we obtain the following recursion for $\varphi(u, y)$:

$$\varphi(u, y) = q\varphi(u + \theta c, y) + p\varphi(u + c - 1, y). \tag{8}$$

Similar to the derivations of (5), after assessing certain ranges of u values, we have

$$\varphi(u, y) - q\varphi(u + \theta c, y) = \begin{cases} p\delta_{u+c-1,-y} & 0 \leq u < 1 - c, \\ p\varphi(u + c - 1, y) & u \geq 1 - c. \end{cases} \tag{9}$$

Let $y = \frac{z}{N}$. Writing $\eta(k|z) = \varphi(\frac{k}{N}, \frac{z}{N})$, (9) can be rewritten as

$$\eta(k|z) - q\eta(k + K_2|z) = \begin{cases} p\delta_{N-K_1-k,z} & 0 \leq k < N - K_1, \\ p\eta(k + K_1 - N|z) & k \geq N - K_1. \end{cases} \tag{10}$$

The above recursive formula can be used to calculate $\varphi(u, y)$ if the initial values $\eta(0|z)$, ..., $\eta(K_2 - 1|z)$ are known for any given z. Next we shall employ the same method as the one used in Section 3 to determine the initial values with the same assumptions regarding N, K_1 and K_2.

Suppose that $nK_2 > K_2 + N - K_1$, then We have

$$\eta(nK_2|z) - \eta(0|z) = \sum_{j=1}^{n} \Big[\eta(jK_2|z) - \eta((j-1)K_2|z)\Big]$$

$$\begin{aligned}
&= p\sum_{j=1}^{J_1} \Big[\eta(jK_2|z) - \delta_{z,(J_1-j+1)K_2}\Big] + p\sum_{j=J_1+1}^{n} \Big[\eta(jK_2|z) - \eta((j-1-J_1)K_2|z)\Big] \\
&= p\sum_{j=1}^{n} \eta(jK_2|z) - p\sum_{j=0}^{n-J_1-1} \eta(jK_2|z) - p\sum_{j=1}^{J_1} \delta_{z,(J_1-j+1)K_2} \\
&= p\sum_{j=n-J_1}^{n} \eta(jK_2|z) - p\eta(0|z) - p\sum_{j=1}^{J_1} \delta_{z,(J_1-j+1)K_2}.
\end{aligned}$$

When $u \to \infty$, $\varphi(u,y)$ tends to zero, so letting $n \to \infty$ the above equation becomes

$$\eta(0|z) = p\eta(0|z) + p\sum_{j=1}^{J_1} \delta_{z,(J_1-j+1)K_2}.$$

Denote i_z and j_z the remainder and quotient from z devided by K_2. One can see that only when $i_z = 0$ and $j_z \le J_1$, in the right-hand side of the above equation, $\sum_{j=1}^{J_1} \delta_{z,(J_1-j+1)K_2} = 1$. Otherwise, $\sum_{j=1}^{J_1} \delta_{z,(J_1-j+1)K_2} = 0$. We can show that, if $K_1 < 2K_2$, i.e. $\theta > 0.5$, then $N - K_2 = N - K_1 + K_1 - K_2 = J_1K_2 + K_1 - K_2$, which leads to $j_z \le J_1$. If $\theta \le 0.5$, then $j_z \le J_1$ is not always satisfied. Therefore, we have the following result

$$\eta(0|z) = \begin{cases} \frac{p}{1-p} & i_z = 0, j_z \le J_1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for $i = 1, 2, \ldots, K_2 - 1$,

$$\eta(nK_2 + i|z) - \eta(i|z) = \sum_{j=1}^{n} \Big[\eta(jK_2 + i|z) - \eta((j-1)K_2 + i|z)\Big]$$

$$\begin{aligned} =\ & p\sum_{j=1}^{J_1} \Big[\eta(jK_2 + i|z) - \delta_{z,(J_1-j+1)K_2+i}\Big] \\ & +p\sum_{j=J_1+1}^{n} \Big[\eta(jK_2 + i|z) - \eta((j-1-J_1)K_2 + i|z)\Big] \\ =\ & p\sum_{j=1}^{n} \eta(jK_2 + i|z) - p\sum_{j=0}^{n-J_1-1} \eta(jK_2 + i|z) - p\sum_{j=1}^{J_1} \delta_{z,(J_1-j+1)K_2+i} \\ =\ & p\sum_{j=n-J_1}^{n} \eta(jK_2 + i|z) - p\eta(i|z) - p\sum_{j=1}^{J_1} \delta_{z,(J_1-j+1)K_2+i}. \end{aligned}$$

When $u \to \infty$, $\varphi(u,y)$ tends to zero, so letting $n \to \infty$ the above equation becomes

$$\eta(i|z) = p\eta(i|z) + p\sum_{j=1}^{J_1} \delta_{z,(J_1-j+1)K_2+i}.$$

Similarly, after considering all possible i_z andj_z values, we obtain, for $i = 1, 2, \ldots, K_2-1$,

$$\eta(i|z) = \begin{cases} \frac{p}{1-p} & i_z = i, j_z \le J_1, \\ 0 & \text{otherwise.} \end{cases}$$

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