

**DEPARTMENT OF  
ACTUARIAL STUDIES  
RESEARCH PAPER  
SERIES**

**Moments of the Accumulation of an  
Annuity under Independent Identically  
Distributed Interest Rates**

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Research Paper No. 2007/04  
October 2007

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# Moments of the Accumulation of an Annuity under Independent Identically Distributed Interest Rates

by Jim Farmer.

## Abstract

Direct expressions are derived for the first four moments of the accumulation of an annuity under the assumption of independent identically distributed interest rates.

## Introduction

Consider the accumulation of an annuity making annual payments in advance for  $n$  years. Assume the interest rates earned over successive years are independent identically distributed random variables, so the accumulation is also a random variable.

An explicit formula for the mean of the accumulation is easily derived. McCutcheon and Scott [1986] provide a recurrence formula which allows the second moment about the origin to be determined and this allows calculation of the variance. This method is included in the CT1 Mathematics of Finance course. Broverman [1991] provides an explicit formula for the second moment, but the derivation of the result calls on a statistical theorem the derivation of which may be beyond the reach of the typical CT1 student. This paper presents an alternative which does not have this problem. This method is extended to provide explicit formulae for the 3<sup>rd</sup> and 4<sup>th</sup> moments.

## Notation

Initially, I'll adopt the notation of McCutcheon and Scott [1986].

The interval of time  $[0, n]$  is subdivided into successive periods  $[0, 1], [1, 2], \dots, [n-1, n]$ , with  $i_t$  denoting the effective interest rate per time period earned in the  $t^{\text{th}}$  time period, which is  $[t-1, t]$ . For ease of explanation, the time period is usually stated to be a year, but the results hold for any general time period.

In this model the  $i_t$  are independent identically distributed random variables with mean  $E(i_t) = j$  and variance  $\text{var}(i_t) = s^2$ .

For  $t = 1, 2, 3, \dots, n$  define  $A_t$  to be the accumulation at time  $t$  of payments of 1 at times  $0, 1, 2, \dots, t-1$ . That is,  $A_t$  is the accumulated value of an annuity in advance with  $t$  payments.

## The Mean

McCutcheon and Scott [1986] provide the now well-known difference equation approach, which uses the equations

$$A_1 = 1 + i_1 \quad (1)$$

$$A_t = (1 + i_t)(1 + A_{t-1}) \quad t = 2, 3, 4, \dots, n. \quad (2)$$

Since  $A_{t-1}$  depends only on  $i_1, i_2, \dots, i_{t-1}$  it is independent of  $i_t$ . Hence (2) allows us to say

$$E(A_t) = E(1 + i_t)E(1 + A_{t-1}) \quad t = 2, 3, 4, \dots, n. \quad (3)$$

which quickly leads to the result

$$E(A_n) = \ddot{s}_{\overline{n}|j} \quad (4)$$

That is, the expected value of the annuity is simply the value of the corresponding annuity using the expected interest rate.

It is interesting to note that an explicit formula for  $A_n$  is

$$A_n = \sum_{t=1}^n \prod_{r=t}^n (1+i_r) \quad (5)$$

and that application of simple expected value properties to this equation also quickly leads to the result of equation (4). However, this approach does not easily generalise to higher moments.

### The Variance

A method for determining the variance from the difference equation is provided by McCutcheon and Scott [1986]. From equation (2) we have

$$E(A_t^2) = E[(1+i_t)^2] E[(1+A_{t-1})^2] \quad t = 2, 3, 4, \dots, n.$$

which is easily developed to give the result

$$E(A_1^2) = 1 + 2j + j^2 + s^2 \quad (6)$$

$$E(A_t^2) = (1 + 2j + j^2 + s^2) \left(1 + 2\ddot{s}_{t-1}^j + E(A_{t-1}^2)\right) \quad t = 2, 3, 4, \dots, n \quad (7)$$

Using equation (6) as a starting point, equation (7) can be used recursively to determine  $E(A_n^2)$ , and then the variance can be determined using

$$\text{var}(A_n) = E(A_n^2) - \{E(A_n)\}^2$$

These equations are easily evaluated by computer. However, if  $n$  is very large, say due to working in weekly time steps over several years, or if the problem is embedded within a simulation, so that it is performed many times for different values of  $j$  and  $s^2$ , then calculation time may be an issue. Hence, it is valuable to note that we can determine an explicit (as opposed to recursive) formula for  $E(A_n^2)$ .

A result has been given by Broverman [1991], who derives it using a result obtained from Feller [1965]. This paper derives the result by an alternative method, solving equation (7) as a linear difference equation. This has two advantages over the method given by Broverman. Firstly, it is more easily generalised to higher moments. Secondly students of the UK actuarial exams encounter this compound interest problem in subject CT1, while the method presented by Broverman may be beyond their grasp until they study subject CT3.

### Solving the difference equation

The value of an annuity with zero payments will be zero, so it is plausible to define  $A_0 = 0$ . Given this definition, equation (7) is also true for  $t = 1$ . Adopting this simpler initial value will simplify later calculations.

For the purposes of solution, there is no need to specify an upper limit on the difference equation. Adopting  $n$  as the dummy variable, equations (6) and (7) can be replaced by

$$E(A_0^2) = 0 \quad (8)$$

$$E(A_n^2) = (1 + 2j + j^2 + s^2) \left(1 + 2\ddot{s}_{n-1}^j + E(A_{n-1}^2)\right) \quad n = 1, 2, 3, \dots \quad (9)$$

For ease of calculation, we also define

$$k = 1 + 2j + j^2 + s^2$$

$$g(n) = E(A_n^2)$$

Then equations (8) and (9) can be written as

$$g(0) = 0 \quad (10)$$

$$g(n) - k g(n-1) = \frac{2k}{j}(1+j)^n - k \left( \frac{2}{j} + 1 \right) \quad (11)$$

The corresponding homogeneous difference equation has characteristic equation

$$x - k = 0$$

and hence has general solution  $g(n) = A k^n$  where  $A$  is an arbitrary constant. The form of the right hand side of equation (11) suggests that we will be able to find a particular solution of the form  $a + b(1+j)^n$  where  $a$  and  $b$  are constants. Substituting this guess into equation (11) gives

$$\begin{aligned} a + b(1+j)^n - k \{ a + b(1+j)^{n-1} \} &= \frac{2k}{j}(1+j)^n - k \left( \frac{2}{j} + 1 \right) \\ a(1-k) + b(1+j-k)(1+j)^{n-1} &= \frac{2k}{j}(1+j)^n - k \left( \frac{2}{j} + 1 \right) \end{aligned}$$

Equate the coefficients of the constant and exponential terms in turn.

$$\begin{aligned} a(1-k) &= -k \left( \frac{2}{j} + 1 \right) \Rightarrow a = \frac{k}{k-1} \left( \frac{2}{j} + 1 \right) \\ b(1+j-k)(1+j)^{n-1} &= \frac{2k}{j}(1+j)^n \Rightarrow b = \frac{2k(1+j)}{j(1+j-k)} \end{aligned}$$

Hence the general solution to equation (11) is

$$g(n) = A k^n + \frac{k}{k-1} \left( \frac{2}{j} + 1 \right) + \frac{2k}{j(1+j-k)} (1+j)^{n+1} \quad (12)$$

To evaluate the arbitrary constant  $A$ , substitute from equation (10).

$$0 = A + \frac{k}{k-1} \left( \frac{2}{j} + 1 \right) + \frac{2k}{j(1+j-k)} (1+j)$$

Substitute the resulting value of  $A$  into equation (12).

$$g(n) = \frac{2k(1+j)}{j\{k-(1+j)\}} \{ k^n - (1+j)^n \} - \frac{k}{k-1} \left( \frac{2}{j} + 1 \right) (k^n - 1)$$

At this point is it of course prudent to check for algebra errors by substituting this result back into equations (10) and (11). That process is routine and is not shown here.

We conclude

$$E(A_n^2) = \frac{2k(1+j)}{j\{k-(1+j)\}} \{ k^n - (1+j)^n \} - \frac{k}{k-1} \left( \frac{2}{j} + 1 \right) (k^n - 1) \quad n = 1, 2, 3, \dots \quad (13)$$

### Comparison to Broverman's result

By comparison, Broverman's approach produces the variance directly, but working backwards from that result shows the second moment about the origin to be:

$$E(A_n^2) = \frac{S^{n+1} - S}{S-1} \times \frac{S+R}{S-R} - \frac{R^{n-1} - R}{R-1} \times \frac{2S}{S-R} \quad (14)$$

where in our notation  $S = k$  and  $R = 1 + j$ . This is a much prettier result than equation (13), particularly when one notes that  $\frac{S^{n+1} - S}{S-1}$  and  $\frac{R^{n-1} - R}{R-1}$  can be written as  $\ddot{s}_{\overline{n}|}$  at the interest rates  $k-1$  and  $j$  respectively. Unfortunately, while this result is very neat, it does not seem to allow easy development of formulae for higher moments, so the later sections of this paper will use a form similar to equation (13).

The right hand sides of equations (13) and (14) can be proved to be equal, but the algebra is omitted since it is tedious and not instructive.

## Higher moments

The process used above to develop equation (13) is easily extended to higher moments. At least it is *conceptually* easy, meaning that the *techniques* required for higher moments are exactly the same as those given above. However, the results do become increasingly complex for higher moments, so the limiting factor when moving to ever higher moments is your ability to perform simple high school algebra to increasingly longer expressions.

In the previous derivation, the prevalence of the terms  $1 + j$  and  $k = 1 + 2j + j^2 + s^2$  suggests that it is useful to define a general symbol for the moments of  $1 + i_t$ .

Define

$$k_r = E\left[(1 + i_t)^r\right]$$

That is,  $k_r$  is the  $r^{\text{th}}$  moment (about the origin) of  $1 + i_t$ . In terms of the previous notation  $k_1 = 1 + j$  and  $k_2$  replaces  $k = 1 + 2j + j^2 + s^2$ .

The three steps for determining the central moments of the annuity are

- Determine the moments (about the origin) of  $1 + i_t$
- Use these to determine the moments (about the origin) of  $A_n$ .
- Use these moments to determine the central moments of  $A_n$ .

Steps 1 and 3 are routine. We will be concerned here with the middle step.

It is also useful to define

$$g_r(n) = E[A_n^r]$$

## First moment revisited

Before launching into the third moment, it is informative to derive the results for the first two moments again but with this new notation. Our starting point is

$$A_0 = 0 \quad (15)$$

$$A_n = (1 + i_n)(1 + A_{n-1}) \quad n = 1, 2, 3, \dots \quad (16)$$

Equation (15) gives

$$g_k(0) = E(A_0^k) = 0 \quad k = 1, 2, 3, \dots \quad (17)$$

From equation (16),

$$E(A_n) = E(1+i_n)E(1+A_{n-1})$$

$$g_1(n) = k_1 \{1 + g_1(n-1)\}$$

yielding the difference equation

$$g_1(n) - k_1 g_1(n-1) = k_1 \quad (18)$$

The corresponding homogeneous difference equation has general solution  $Ak_1^n$  where  $A$  is an arbitrary constant.

The right hand side suggests there may be a constant particular solution,  $g_1(n) = a$ . Substituting this guess in equation (18) gives:

$$a - k_1 a = k_1$$

$$a = \frac{k_1}{1 - k_1}$$

Thus the general solution is

$$g_1(n) = Ak_1^n - \frac{k_1}{k_1 - 1} \quad (19)$$

Equation (17) implies  $g_1(0) = 0$ , which gives

$$0 = A - \frac{k_1}{k_1 - 1}$$

Substitute the resulting value of  $A$  in (19).

$$g_1(n) = \frac{k_1}{k_1 - 1} (k_1^n - 1) \quad (20)$$

Substituting  $k_1 = 1 + j$  shows that this result is  $\ddot{s}_{\overline{n}|j}$  at  $j$ , as found previously.

## Second moment revisited

From equation (16),

$$E(A_n^2) = E[(1+i_n)^2]E[(1+A_{n-1})^2]$$

$$g_2(n) = k_2 \{1 + 2E(A_{n-1}) + E(A_{n-1}^2)\}$$

$$= k_2 \{1 + 2g_1(n-1) + g_2(n-1)\}$$

Using the result of (20) gives the difference equation

$$g_2(n) - k_2 g_2(n-1) = k_2 \left\{ 1 + \frac{2k_1}{k_1 - 1} (k_1^{n-1} - 1) \right\} \quad (21)$$

The corresponding homogeneous difference equation has general solution  $Ak_2^n$  where  $A$  is an arbitrary constant.

The right hand side suggests there may be a particular solution of the form  $g_2(n) = a + bk_1^n$ . Substituting this guess in equation (21) gives:

$$a + bk_1^n - k_2(a + bk_1^{n-1}) = k_2 \left\{ 1 + \frac{2k_1}{k_1 - 1} (k_1^{n-1} - 1) \right\}$$

Equate the constant and exponential components.

$$a - k_2 a = k_2 \left\{ 1 - \frac{2k_1}{k_1 - 1} \right\} \Rightarrow a = \frac{k_2}{k_2 - 1} \left\{ \frac{2k_1}{k_1 - 1} - 1 \right\}$$

$$bk_1^n - k_2 bk_1^{n-1} = k_2 \left\{ \frac{2k_1}{k_1 - 1} k_1^{n-1} \right\} \Rightarrow b = -\frac{2k_1 k_2}{(k_1 - 1)(k_2 - k_1)}$$

Thus the general solution is

$$g_2(n) = Ak_2^n + \frac{k_2}{k_2 - 1} \left\{ \frac{2k_1}{k_1 - 1} - 1 \right\} - \frac{2k_1 k_2}{(k_1 - 1)(k_2 - k_1)} k_1^n \quad (22)$$

Equation (17) implies  $g_2(0) = 0$ , which gives

$$0 = A + \frac{k_2}{k_2 - 1} \left\{ \frac{2k_1}{k_1 - 1} - 1 \right\} - \frac{2k_1 k_2}{(k_1 - 1)(k_2 - k_1)}$$

Substitute the resulting value of  $A$  in (22).

$$g_2(n) = \frac{2k_1 k_2}{(k_1 - 1)(k_2 - k_1)} (k_2^n - k_1^n) - \frac{k_2}{k_2 - 1} \left( \frac{2k_1}{k_1 - 1} - 1 \right) (k_2^n - 1) \quad (23)$$

It is easily shown that this result is equivalent to equation (13).

## The Useful Pattern

We could develop the formula for the 3<sup>rd</sup> moment using exactly the same approach as given above. In fact, I did, and after doing so a useful pattern became obvious. The pattern is perhaps not *obvious* from the formula for the 2<sup>nd</sup> moment, but it is detectable if you know where to look, so there is no reason why we should not take advantage of it now.

In the previous section we found the solution to the equations

$$g_2(0) = 0$$

$$g_2(n) - k_2 g_2(n-1) = k_2 \left\{ 1 + \frac{2k_1}{k_1 - 1} (k_1^{n-1} - 1) \right\}$$

to be

$$g_2(n) = \frac{2k_1 k_2}{(k_1 - 1)(k_2 - k_1)} (k_2^n - k_1^n) - \frac{k_2}{k_2 - 1} \left( \frac{2k_1}{k_1 - 1} - 1 \right) (k_2^n - 1)$$

The difference equation can be written in the form

$$g_2(n) - k_2 g_2(n-1) = k_2 \{ B + Ck_1^{n-1} \}$$

where  $B$  and  $C$  are constants, and the solution is

$$g_2(n) = \frac{k_2}{k_2 - 1} B (k_2^n - 1) + \frac{k_2}{k_2 - k_1} C (k_2^n - k_1^n)$$



The same pattern holds, in a somewhat trivial form, for  $g_1(n)$ . So, we might guess that it will also hold for higher moments. We'll guess that the difference equations for the 3<sup>rd</sup> moment will take the form:

$$g_3(0) = 0 \quad (24)$$

$$g_3(n) - k_3 g_3(n-1) = k_3 \{B + Ck_1^{n-1} + Dk_2^{n-1}\} \quad (25)$$

In the next section we'll verify this to be the case. We'll guess that these equations have the solution

$$g_3(n) = \frac{k_3}{k_3-1} B(k_3^n - 1) + \frac{k_3}{k_3-k_1} C(k_3^n - k_1^n) + \frac{k_3}{k_3-k_2} D(k_3^n - k_2^n) \quad (26)$$

We can easily verify that this is a valid solution. Substituting  $n = 0$  in equation (26) gives

$$g_3(0) = \frac{k_3}{k_3-1} B(1-1) + \frac{k_3}{k_3-k_1} C(1-1) + \frac{k_3}{k_3-k_2} D(1-1) = 0$$

as required by equation (24). Substituting equation (26) into equation (25) produces

$$\begin{aligned} & g_3(n) - k_3 g_3(n-1) \\ &= \frac{k_3}{k_3-1} B(k_3^n - 1) + \frac{k_3}{k_3-k_1} C(k_3^n - k_1^n) + \frac{k_3}{k_3-k_2} D(k_3^n - k_2^n) \\ &\quad - k_3 \left\{ \frac{k_3}{k_3-1} B(k_3^{n-1} - 1) + \frac{k_3}{k_3-k_1} C(k_3^{n-1} - k_1^{n-1}) + \frac{k_3}{k_3-k_2} D(k_3^{n-1} - k_2^{n-1}) \right\} \\ &= \frac{k_3}{k_3-1} B(k_3^n - 1) + \frac{k_3}{k_3-k_1} C(k_3 k_1^{n-1} - k_1^n) + \frac{k_3}{k_3-k_2} D(k_3 k_2^{n-1} - k_2^n) \\ &= k_3 \{B + Ck_1^{n-1} + Dk_2^{n-1}\} \end{aligned}$$

as required by equation (25). This shows equation (26) to be a valid solution to equations (24) and (25). Also, the manner in which the working above plays out gives confidence that this pattern of solution will continue to work for higher moments.

### The third moment

From equation (16),

$$\begin{aligned} E(A_n^3) &= E\left[(1+i_n)^3\right] E\left[(1+A_{n-1})^3\right] \\ g_3(n) &= k_3 \left\{ 1 + 3E(A_{n-1}) + 3E(A_{n-1}^2) + E(A_{n-1}^3) \right\} \\ &= k_3 \left\{ 1 + 3g_1(n-1) + 3g_2(n-1) + g_3(n-1) \right\} \end{aligned}$$

Using the results of (20) and (23) gives the difference equation

$$g_3(n) - k_3 g_3(n-1) = k_3 \left\{ \begin{aligned} & 1 + \frac{3k_1}{k_1-1} (k_1^{n-1} - 1) + \frac{6k_1 k_2}{(k_1-1)(k_2-k_1)} (k_2^{n-1} - k_1^{n-1}) \\ & - \frac{3k_2}{k_2-1} \left( \frac{2k_1}{k_1-1} - 1 \right) (k_2^{n-1} - 1) \end{aligned} \right\}$$

which can be developed to

$$g_3(n) - k_3 g_3(n-1) = k_3 \left[ \begin{aligned} &1 - \frac{3k_1}{k_1-1} - \frac{3k_2}{k_2-1} + \frac{6k_1k_2}{(k_1-1)(k_2-1)} \\ &+ \left\{ \frac{3k_1}{k_1-1} - \frac{6k_1k_2}{(k_1-1)(k_2-k_1)} \right\} k_1^{n-1} \\ &+ \left\{ \frac{3k_2}{k_2-1} - \frac{6k_1k_2}{(k_1-1)(k_2-1)} + \frac{6k_1k_2}{(k_1-1)(k_2-k_1)} \right\} k_2^{n-1} \end{aligned} \right] \quad (27)$$

Also, equation (17) gives  $g_3(0) = 0$ . So we can use the pattern from the previous section to immediately state the solution as

$$\begin{aligned} g_3(n) &= \frac{k_3}{k_3-1} \left\{ 1 - \frac{3k_1}{k_1-1} - \frac{3k_2}{k_2-1} + \frac{6k_1k_2}{(k_1-1)(k_2-1)} \right\} (k_3^n - 1) \\ &+ \frac{k_3}{k_3-k_1} \left\{ \frac{3k_1}{k_1-1} - \frac{6k_1k_2}{(k_1-1)(k_2-k_1)} \right\} (k_3^n - k_1^n) \\ &+ \frac{k_3}{k_3-k_2} \left\{ \frac{3k_2}{k_2-1} + \frac{6k_1k_2}{(k_1-1)(k_2-k_1)} - \frac{6k_1k_2}{(k_1-1)(k_2-1)} \right\} (k_3^n - k_2^n) \end{aligned} \quad (28)$$

### Formulae for efficient calculation

Clearly the complexity of the higher moments increases, and if we continue in this manner it may be difficult to keep the comparable expression for the 4<sup>th</sup> moment within the confines of the page.

Also, it is important to not lose sight of the original problem. We originally noted that for the second moment we can simply use equation (6) as a starting point and applying equation (7) recursively. Similar formulae can be found higher moments. The scenarios which justify preferring explicit formulae such as (28) over recursive formulae are those where calculation time is an issue. If calculation time is an issue, it isn't sufficient to provide an equation like (28), which clearly contains some repeating terms. Rather, we should aim to specify a method of calculation which minimises unnecessary repetition of calculations.

Here is one method which seems reasonably efficient.

We have previously found

$$g_1(n) = \frac{k_1}{k_1-1} (k_1^n - 1)$$

This can be written as

$$g_1(n) = P_0 + P_1 k_1^n \quad (29)$$

where

$$\begin{aligned} P_0 &= \frac{k_1}{1-k_1} \\ P_1 &= -P_0 \end{aligned} \quad (30)$$

The use of the symbol  $P_1$  seems superfluous, but its use aids the recognition of a pattern at the higher moments.

We then found

$$\begin{aligned}
g_2(n) &= k_2 \{1 + 2E(A_{n-1}) + E(A_{n-1}^2)\} \\
g_2(n) - k_2 g_2(n-1) &= k_2 \{1 + 2g_1(n-1)\} \\
&= k_2 \{1 + 2(P_0 + P_1 k_1^{n-1})\} \\
&= k_2 \{(1 + 2P_0) + 2P_1 k_1^{n-1}\}
\end{aligned}$$

Hence

$$\begin{aligned}
g_2(n) &= \frac{k_2}{k_2 - 1} (1 + 2P_0) (k_2^n - 1) + \frac{k_2}{k_2 - k_1} 2P_1 (k_2^n - k_1^n) \\
&= -\left(\frac{k_2}{k_2 - 1}\right) (1 + 2P_0) - \left(\frac{k_2}{k_2 - k_1}\right) 2P_1 k_1^n \\
&\quad + \left\{ \left(\frac{k_2}{k_2 - 1}\right) (1 + 2P_0) + \left(\frac{k_2}{k_2 - k_1}\right) 2P_1 \right\} k_2^n
\end{aligned}$$

This can be written

$$g_2(n) = Q_0 + Q_1 k_1^n + Q_2 k_2^n \quad (31)$$

where

$$\begin{aligned}
Q_0 &= \frac{k_2}{1 - k_2} (1 + 2P_0) \\
Q_1 &= \frac{k_2}{k_1 - k_2} 2P_1 \\
Q_2 &= -(Q_0 + Q_1)
\end{aligned} \quad (32)$$

For the 3<sup>rd</sup> moment we found

$$\begin{aligned}
g_3(n) &= k_3 \{1 + 3E(A_{n-1}) + 3E(A_{n-1}^2) + E(A_{n-1}^3)\} \\
g_3(n) - k_3 g_3(n-1) &= k_3 \{1 + 3g_1(n-1) + 3g_2(n-1)\} \\
&= k_3 \{1 + 3(P_0 + P_1 k_1^{n-1}) + 3(Q_0 + Q_1 k_1^{n-1} + Q_2 k_2^{n-1})\} \\
&= k_3 \{(1 + 3P_0 + 3Q_0) + (3P_1 + 3Q_1) k_1^{n-1} + 3Q_2 k_2^{n-1}\}
\end{aligned}$$

Hence

$$\begin{aligned}
g_3(n) &= \frac{k_3}{k_3 - 1} (1 + 3P_0 + 3Q_0) (k_3^n - 1) + \frac{k_3}{k_3 - k_1} (3P_1 + 3Q_1) (k_3^n - k_1^n) \\
&\quad + \frac{k_3}{k_3 - k_2} 3Q_2 (k_3^n - k_2^n) \\
&= -\frac{k_3}{k_3 - 1} (1 + 3P_0 + 3Q_0) - \left(\frac{k_3}{k_3 - k_1}\right) (3P_1 + 3Q_1) k_1^n - \left(\frac{k_3}{k_3 - k_2}\right) 3Q_2 k_2^n \\
&\quad + \left\{ \frac{k_3}{k_3 - 1} (1 + 3P_0 + 3Q_0) + \frac{k_3}{k_3 - k_1} (3P_1 + 3Q_1) + \frac{k_3}{k_3 - k_2} 3Q_2 \right\} k_3^n
\end{aligned}$$

This can be written

$$g_3(n) = R_0 + R_1 k_1^n + R_2 k_2^n + R_3 k_3^n \quad (33)$$

where

$$\begin{aligned}
R_0 &= \frac{k_3}{1-k_3}(1+3P_0+3Q_0) \\
R_1 &= \frac{k_3}{k_1-k_3}(3P_1+3Q_1) \\
R_2 &= \frac{k_3}{k_2-k_3}3Q_2 \\
R_3 &= -(R_0+R_1+R_2)
\end{aligned} \tag{34}$$

Equations (29) to (34) provide an efficient means of calculating the 3<sup>rd</sup> central moment. If the 4<sup>th</sup> moment is required, a similar process gives:

$$g_4(n) = S_0 + S_1k_1^n + S_2k_2^n + S_3k_3^n + S_4k_4^n \tag{35}$$

where

$$\begin{aligned}
S_0 &= \frac{k_4}{1-k_4}(1+4P_0+6Q_0+4R_0) \\
S_1 &= \frac{k_4}{k_1-k_4}(4P_1+6Q_1+4R_1) \\
S_2 &= \frac{k_4}{k_2-k_4}(6Q_2+4R_2) \\
S_3 &= \frac{k_4}{k_3-k_4}4R_3 \\
S_4 &= -(S_0+S_1+S_2+S_3)
\end{aligned} \tag{36}$$

It could be argued this solution is recursive, since to determine the 4<sup>th</sup> moment,  $g_4(n)$ , we first determine the first 3 moments. So it may be asked why this solution is any better than determining the 4<sup>th</sup> moment version of equations (6) and (7), and applying them recursively. The answer to this is that equations (6) and (7) are recursive on  $n$ , which is potentially a very large positive integer. By contrast, the above method for determining  $g_r(n)$  is only recursive on  $r$ , and it is unlikely we will ever need moments beyond  $g_4(n)$ , the 4<sup>th</sup> moment.

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