Collusion, price dispersion, and fringe competition*

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Abstract

We study the optimal behaviour of a cartel faced with fringe competition and imperfectly attentive consumers. Intertemporal price dispersion obfuscates consumer price comparison which aids the cartel through two channels: it reduces the effectiveness of free riding by the fringe; and it relaxes the cartel’s internal incentive constraints. Our theory provides a collusive rationale for sales and Edgeworth cycles, explains the survival of a price-setting cartel in a homogeneous product market, and characterises the cartel’s manipulation of its fringe rival through a simple cut-off rule.

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Repeated interaction between firms is ubiquitous, and industries are often concentrated. The theory of collusion therefore has broad relevance for industry interaction. Theory typically presumes full membership of a cartel. This presumption is important because incomplete cartels are vulnerable to fringe competition. Consider an industry characterised by simultaneous price competition for a homogeneous product under perfect information. Standard arguments establish a spectrum of profitable equilibria for a patient cartel with complete membership. However, profitable equilibria are not available for an incomplete cartel. Documented cartels also reveal an important role for fringe competition. The international Vitamin C cartel, operating from 1991 to 1995, was ultimately destroyed by the persistent price undercutting of Chinese firms operating outside the cartel (de Roos, 2001). The Vitamin B1 cartel also suffered substantial market share losses at the hands of excluded

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While fringe competition is often neglected, the problem of a cartel or dominant firm subject to fringe competition has a long tradition. Early examples include Markham (1951) and Arant (1956).
Chinese competitors (Bos and Harrington, 2010). The difficulties faced by the OPEC cartel due to incomplete membership are well known.

As Bos and Harrington (2010) argue, incomplete membership is probably the norm for explicit cartels.\footnote{Additional details can be found in the cross-cartel studies of Hay and Kelley (1974) and Griffin (1989) and the recent case evidence of Harrington (2006).} Moving beyond explicit cartels, industries are often concentrated, but rarely completely controlled by a small number of firms. To illustrate, based on the 2007 U.S. Census, over 23 per cent of industries have an 8-firm concentration above 0.5 and a 20-firm concentration lower than 0.9.\footnote{To calculate concentrations, we use an 8-digit NAICS-based classification, omitting mining, construction, and manufacturing industries. Using the same source, over 13 per cent of industries have a 4-firm concentration above 0.5 and a 20-firm concentration below 0.9.} Whether considering settings of explicit or implicit coordination, theory must therefore account for the potentially irritating presence of peripheral firms. We contribute by considering a price-setting cartel operating in a homogeneous product market, impinged by a strategic fringe.

In our model, a cartel engages in infinitely repeated price competition with a strategic, but statically optimising player, which we label the \textit{fringe}. Consumers are imperfectly attentive in the manner of de Roos and Smirnov (2013). Consumers are better able to recognise price discrepancies when the price process is simple and transparent. In particular, if prices are fixed over time, price differences are easy to discern; while dispersed price patterns are less amenable to price comparison. In this setting, the cartel strategically employs a dispersed price path to obfuscate price comparison efforts of consumers.

The combination of homogeneous products and imperfect consumer attentiveness gives rise to two contrasting temptations. With identical products, cartel members have an incentive to \textit{undercut} the cartel price to obtain a discrete increase in market share. Alternatively, a cartel member who sets a higher price than her peers may still attract unwitting customers, providing an incentive to \textit{relent}. Cartel policies must therefore guard against both temptations, giving rise to two distinct sets of incentive constraints. The fringe is beset by similar motives, and it is here that the cartel is able to manipulate the fringe. By crafting the path of prices, the cartel can induce the fringe to either undercut the cartel price or relent.

We characterise the optimal dynamic price path of the cartel within the class of infinitely repeated sequences of finite length.\footnote{Restriction to finite-complexity paths follows from two alternative motivations. First, difficulties in coordinating, negotiating, and enforcing cartel policies likely rise with the complexity of the price path. A trade-off between coordination difficulties and profitability could give rise to an optimal cycle complexity. Second, as de Roos and Smirnov (2013) show in a related context, finite-length paths also emerge endogenously if consumers have finite memory.} Our first result highlights the value of price dispersion for the cartel. No fixed price path is sustainable in the presence of the fringe. Without price variation, price differences are obvious to consumers and the fringe has an incentive to undercut the cartel price and capture the entire market. By introducing price variation, the cartel reduces consumer responsiveness to price differences, ensuring its survival despite undercutting by the fringe.

The optimal path is weakly decreasing and is described by a set of complementary slackness conditions. In each period of a sequence, cartel members either set the monopoly price
or decrease prices. Price dispersion is most efficiently achieved through sales, allowing the cartel to enjoy monopoly prices for extended periods, interrupted by temporary price reductions. The most robust structure to deter internal deviation is an Edgeworth cycle in which the cartel reduces price in each period of a sequence.

The cartel is able to manipulate the fringe by adjusting the level of prices in each period. High prices encourage the fringe to undercut, while low prices induce relenting. The cartel thus faces a discrete trade-off between markup and market share. We show that cartel manipulation of the fringe takes the form of a simple cut-off rule. In each finite price sequence, the fringe undercut the monopoly price observed in the first period, and the cartel decides in which period the fringe should begin relenting.

Our principal insight is that obfuscation (in the form of intertemporal price dispersion) aids collusion through two mechanisms. First, it reduces the market share of a fringe choosing to undercut the cartel price, mitigating the effectiveness of free riding by the fringe and allowing the survival of the cartel. Second, it eases the cartel's internal incentive constraints by reducing the market share of a deviator contemplating undercutting the cartel price. This insight has potential implications for both tacit and explicit collusion. In the presence of explicit communication, our focus on the optimal collusive mechanism is natural. This leads to predictions about the shape of the optimal path and the nature of fringe manipulation. Without explicit communication, we expect coordination on the optimal path to be more challenging. However, our main message likely still applies. Dispersed prices reduce the incentive for major firms to rock the boat and mitigate the disruptive influence of peripheral firms.

We contribute to a body of work examining cartel stability in the presence of fringe competition. Given the vulnerability of a cartel in homogeneous-good price-setting environments, much of the focus has been on a quantity setting or capacity constrained fringe. Cartel stability in homogeneous product markets has been examined in the context of a price-taking fringe (e.g. D'Aspremont et al. (1983); Donsimoni et al. (1986)), and a Cournot fringe (e.g. Shaffer (1995); Martin (1990)). D’Aspremont and Gabszewicz (1985), Lambson (1994), and Bos and Harrington (2010) consider a fringe setting prices in a homogeneous product market. Cartel survival is aided by capacity constraints on the fringe. de Roos (2001) analyses the dynamic destruction of a cartel faced with a free riding fringe in a homogeneous product market. In our setting, it is the imperfect attentiveness of consumers that allows cartel survival despite a homogeneous product environment.

We build directly on the framework of de Roos and Smirnov (2013) by introducing a fringe firm into the problem of a cartel facing imperfectly attentive consumers. In de Roos and Smirnov (2013), consumer inattention improves the sustainability of a cartel willing to craft a dispersed price path. We reinforce this finding by introducing a second motive for price variation: price dispersion not only relaxes the cartel’s internal incentive constraints, but also weakens the destructive power of the fringe.

The rest of the paper is structured as follows. In Section 1, we provide two illustrative examples. In Section 2, we introduce the model and describe the problem faced by the cartel and the fringe. Section 3 contains our main results. We first analyse the cartel problem
under the assumption that the fringe primarily engages in undercutting. We then examine the manner in which the cartel manipulates the fringe. Sections 4 and 5 contain a discussion and conclusion.

1 Preliminary examples

In this section, we introduce two motivating examples. We first lay out some features common to both examples. $n$ firms, indexed by $j$, produce identical products and compete by simultaneously setting price over an infinite horizon. A unit mass of consumers each wishes to buy a single unit of the good. Consumers have a reservation price of 1. Consumer beliefs about prices are summarised by the pricing bounds $p$ and $\overline{p}$. If $\min\{p_j\}_{j=1}^n < p$, then consumers are surprised to see such a low price and all consumers buy from the cheapest firm. Similarly, if $\max\{p_j\}_{j=1}^n > \overline{p}$, consumers notice the unusually high price and avoid the most expensive firm. However, consumers pay less attention to prices that fall within their expectations. Specifically, if $p_j \in [p, \overline{p}]$, $j = 1, \ldots, n$, then the market shares of the lowest to highest priced firms are given by $\alpha_1, \ldots, \alpha_n$, respectively, where $0 < \alpha_n < \cdots < \alpha_1 < 1$ and $\sum_{j=1}^n \alpha_j = 1$.

A market equilibrium imposes two requirements. First, consumer beliefs are consistent with equilibrium play. Below, we illustrate two alternative definitions of consistency. Second, firm strategies are subgame perfect.

In Section 1.1, we consider a dominant firm battling a fringe competitor. We illustrate the value of price dispersion in combating the fringe, and the construction of a price path to manipulate the behaviour of the fringe. Section 1.2 introduces the problem of a cartel faced with a fringe rival. We highlight the dual role of price dispersion in combating the fringe and relaxing the cartel’s internal incentive constraints.

For reference, define the following time-varying profile of prices:

$$p(t) = \begin{cases} r^1 & \text{if } t = 0 \text{ or } t \text{ is even;} \\ r^2 & \text{if } t \text{ is odd.} \end{cases}$$

(1)

1.1 A dominant firm and fringe

The market consists of two firms. Firm 1 is an intertemporal optimiser with discount factor $\delta$. Firm 2 is a myopic profit maximiser, and we attach the label fringe to Firm 2. We impose the following consistency requirement on market equilibrium: $p$ and $\overline{p}$ are equal to the lowest and highest prices on the equilibrium path, respectively.

First, suppose that as part of a market equilibrium, Firm 1 sets a constant price, $p_1$. Consider two alternatives for the fringe. If the fringe marginally undercut $p_1$, it could capture the entire market. Alternatively, if the fringe sets a price $p_2 > p_1$, then $p = p_1$ and $\overline{p} = p_2$. The fringe would then earn profits of $\alpha_2 p_2$. In this case, the fringe optimally chooses the reservation price, $p_2 = 1$, yielding profits of $\alpha_2$. Suppose $p_1 > \alpha_2$. The fringe therefore prefers to
undercut, leaving Firm 1 with no customers and no profits, and leading to a contradiction. Thus, there is no equilibrium in which Firm 1 sets a constant price \( p_1 > \alpha_2 \). Suppose instead that \( p_1 \leq \alpha_2 \). The fringe prefers to set the reservation price \( p_2 = 1 \). Firm 1 then optimally chooses \( p_1 = \alpha_2 \), and these choices form the basis of a market equilibrium.

Next, suppose that Firm 1 adopts the price profile (1) with \( r_1 = 1 \) and \( r_2 = \alpha_2 \). Suppose that in equilibrium, \( p = \alpha_2 \) and \( \overline{p} = 1 \). Then, the fringe has an incentive to marginally undercut Firm 1 when \( p_1 = 1 \), obtaining profits of approximately \( \alpha_1 \); and an incentive to set the reservation price when \( p_1 = \alpha_2 \), obtaining profits of \( \alpha_2 \). With these price profiles, we can confirm that \( p = \alpha_2 \) and \( \overline{p} = 1 \). Next, note that there is no price profile of the form (1) that improves on this strategy. If Firm 1 were to set a price \( p_1 > \alpha_2 \) in odd periods, then the fringe would undercut and capture the entire market. The price \( p_1 = \alpha_2 \) is the highest price that manipulates the fringe into raising price. In even periods, the fringe will undercut any price Firm 1 sets. It is therefore optimal to set the reservation price.

### 1.2 A cartel and fringe

The market consists of three firms. Firms \( j = 1, 2 \) operate a price fixing cartel and have common discount factor \( \delta \). Firm \( j = 3 \) acts as a fringe, operating outside the cartel and behaving myopically. If \( p_j \in [p, \overline{p}] \), \( j = 1, 2, 3 \), then the market shares of the lowest to highest priced firms are given by \( \alpha_1 = 7/16 \), \( \alpha_2 = 5/16 \), and \( \alpha_3 = 1/4 \), respectively.

In the previous example, we identified an equilibrium in which the fringe firm was systematically more expensive than her rival. One might suspect that consumers gradually learn about persistent price differences. We therefore modify the definition of consistency in market equilibrium for this example. In particular, we assign \( p \) and \( \overline{p} \) as the lowest and highest prices set by at least two firms on the equilibrium path, respectively.

Consider the following class of grim-trigger strategies for the cartel. Given a collusive path, \( \{p(t)\}_{t=0}^{\infty} \), define the grim-trigger strategy

\[
p_j^t = \begin{cases} 
p(t) & \text{if } t = 0 \text{ or } p_i^\tau = p(\tau) \text{ for } \tau < t \text{ and } i = 1, 2; \\
0 & \text{otherwise.}
\end{cases}
\]

Specifically, define two grim trigger strategies: \( \sigma^1 \) is a grim-trigger strategy with \( p(t) = r_1 \forall t \); and \( \sigma^2 \) is a grim-trigger strategy with \( p(t) \) given by the time-varying profile in (1). Under \( \sigma^1 \), firms set a constant price on the equilibrium path, while \( \sigma^2 \) involves a two-period cycle. Let us consider the sustainability of both strategies when \( r_1 = 1 \) and \( r_2 = \alpha_3 = 1/4 \).

First, observe that if \( \sigma^1 \) describes cartel strategies in a market equilibrium, then \( p = \overline{p} = 1 \). In our setting, with no price variation, consumers are accustomed to seeing the same price and any departure is salient to all consumers. With a myopic perspective, the fringe firm has an incentive to undercut the cartel price and capture the entire market. This is true for any \( r^1 > 0 \), and a profitable fixed-price cartel is not sustainable for any discount factor.\(^5\)

\(^5\)Notice that our definition of the price reference range plays an important role here. If we define \( p \) and \( \overline{p} \) as the lowest and highest prices set on the equilibrium path and cartel members are sufficiently patient, then an equilibrium exists in which the fringe persistently sets a price higher than the cartel.
Next, consider the sustainability of $\sigma^2$ as part of a market equilibrium. Under $\sigma^2$, $p = \alpha_3$ and $\bar{p} = 1$. The viability of $\sigma^2$ must be evaluated in two subgames: the subgames beginning at the peak and trough of each cycle in the cartel’s price path. Let us first examine the behaviour of the fringe. At the peak of the cycle, the best response of the fringe is to marginally undercut the cartel price of 1 and obtain market share $\alpha_1$. At the trough of the cycle, the fringe could marginally undercut the cartel price of $\alpha_3$ and capture the whole market; alternatively, the fringe could raise price to $r^1 = 1$ and obtain market share of $\alpha_3$. By choosing $r^2 = \alpha_3$, the cartel can manipulate the fringe into raising price at the cycle trough.

Let $v^1$ be the value of $\sigma^2$ to cartel members at the cycle peak; then, $v^1 = \frac{1 - \alpha_1 + \alpha_3 (1 - \alpha_3) \delta}{2 (1 - \delta^2)}$. At the cycle peak, a deviating firm can marginally undercut both its collaborator and the fringe to deliver a profit approaching $\alpha_1$. This deviation is unprofitable if and only if $\delta \geq 1/2$. A potential deviator may also be tempted to lower price below $p$ to capture the whole market. With $p = \alpha_3$, this deviation is also not profitable if $\delta \geq 1/2$.

Next, let $v^2$ be the value of $\sigma^2$ to cartel members at the trough. Hence, $v^2 = \frac{\alpha_3 (1 - \alpha_3) + (1 - \alpha_1) \delta}{2 (1 - \delta^2)}$. Marginally undercutting gives profits approaching $\alpha_3$, which is not profitable for $\delta \geq 1/2$. A final deviation is worth considering. Because consumers are imperfectly attentive, a firm setting a higher price than her rival still attracts a positive market share. The optimal deviation of this kind involves marginally undercutting the price of the fringe, yielding profits approaching $\alpha_2$. For $\delta \geq 1/2$ this is also unattractive. Consequently, the strategy $\sigma^2$ is sustainable for $\delta \geq 1/2$. For $\delta \in [1/2, 1]$, $\sigma^2$ is sustainable while $\sigma^1$ is not.

2 The model

A market comprises a set of $n$ firms, indexed by $j = 1, \ldots, n$. Firms sell an undifferentiated product to a unit mass of consumers in each time period $t = 0, \ldots, \infty$. Each consumer has a downward-sloping demand function given by $D(p)$ with $\int_0^\infty D(x) \, dx < \infty$, and associated differentiable revenue function $R(p) = pD(p)$. Consumers are not strategic players; they myopically maximise surplus in the current period. $n - 1$ firms participate in a price-setting cartel. The remaining firm $f = n$ acts as the fringe. Cartel members discount the future at the common rate $\delta \in (0, 1)$, while the fringe firm behaves myopically ($\delta = 0$). All firms have constant marginal costs which we normalise to zero. Assumption 1 places mild restrictions on the demand function. For future reference, we will refer to the price $\hat{p}$ as the monopoly price.

Assumption 1. The revenue function $R(p)$ is differentiable, attains a unique maximum at price $\hat{p}$, and satisfies $R'(p) > 0$ for $p < \hat{p}$.

Consumer price information is imperfect. Consumers do not observe and recall specific prices unless they are unusual or attractive. Rather, consumers have an impression of the price distribution. The following market share function summarises consumer behaviour. We define market shares in terms of the share of consumers rather than output. Suppose consumer beliefs about the lower and upper bounds of the support of the price distribution
are given by \( p \) and \( \overline{p} \), respectively. Let \( p_{(m)} \) be the \( m \)th lowest price, and \( \mathbf{p} \equiv (p_1, p_2, \ldots, p_n) \) denote a price vector. The market share for firm \( j \) is then given by\(^6\)

\[
\alpha_{m} \left\{ \begin{array}{ll}
0, & \text{if } p_{(1)} < \min\{p_j, \overline{p}\} \text{ or } p_j > \max\{p_{(1)}, \overline{p}\} \\
\left(\sum_{i=m}^{n} \alpha_i\right) / s & \text{if } p_j = p_{(m)} \text{ and } p_i \in [p_j, \overline{p}] \forall i \text{ and } p_j \neq p_i \forall i \neq j \\
1, & \text{if } p_j < \min\{p_j, p_{(2)}\} \text{ or } p_{(2)} > \max\{p_j, \overline{p}\},
\end{array} \right.
\]

(3)

where \( \alpha_m \) is the market share enjoyed by the firm setting the \( m \)th lowest price, \( \alpha_i \geq \alpha_{i+1} \) for \( i = 1, \ldots, n-1 \), and \( \sum_{i=1}^{n} \alpha_i = 1 \). If all the firms set the same price, the market is shared equally. If prices differ, market shares depend on the relationship between the price vector and the consumer reference bounds \( p \) and \( \overline{p} \). A firm setting a price lower than all of her rivals captures the entire market if her price is outside the range \([p, \overline{p}]\); but she attracts a market share of \( \alpha_1 \in (1/n, 1) \) if her price is inside this range. Intuitively, if she sets a price that accords with consumer experience, she does not attract the attention of all consumers and she may receive a market share less than one. Similarly, a firm setting a price higher than her peers attracts no customers if her price is outside the reference bounds, but obtains a market share \( \alpha_n \in (0, 1/n) \) if her price is inside the reference bounds. The firm setting the \( m \)th lowest price obtains the \( m \)th highest market share, \( \alpha_m \). For future reference, define \( \alpha \) as the vector of market share parameters, \( \alpha = \{\alpha_i\}_{i=1}^{n} \).

**Assumption 2.** Market shares, \( s(p, p, \overline{p}) \), are given by (3).

Consumer attentiveness may vary systematically across markets, depending on, for example, the frequency of purchase, importance of a product in the consumer budget, and prominence of price displays. Accordingly, the market share parameters \( \alpha_m, m = 1, \ldots, n \), will vary across markets. For instance, we may expect the market share of the lowest priced firm, \( \alpha_1 \), to depend positively on consumer attentiveness, while there is a negative relationship between attentiveness and the share of the highest priced firm, \( \alpha_n \).

The market share function (3) has two natural interpretations. First, it represents a minor adjustment to the Bertrand rationing rule, incorporating the possibility of consumer inattentiveness. Second, we could see it as a reduced form of (say) a consumer search process. Example 1 offers a simple illustration of consumer behaviour consistent with (3). The consumer search model of de Roos and Smirnov (2013) is also consistent with (3).

**Example 1.** Consider the following setting. If a store sets a price outside the range \([p, \overline{p}]\), it attracts the attention of all consumers. Otherwise, each consumer samples the store with independent probability \( \beta \). Consumers then choose the lowest price of all the prices they have observed. If a consumer samples no stores, she randomises between each store with

\(^6\)Strictly speaking, (3) is incomplete. In particular, price vectors for which a subset of prices \( p_j \in [p, \overline{p}] \) and there exists at least one \( i \) such that \( p_i > \overline{p} \) are not all specified. Such price profiles will not be observed on the equilibrium path. Extension to cover these profiles is straightforward.
probability $1/n$. If all prices are within the range $[p, \overline{p}]$, then the market share of the store with the $m^{th}$ lowest price is

$$\alpha_m = (1 - \beta)^n \frac{1}{n} + (1 - \beta)^{m-1} \beta,$$

$m = 1, \ldots, n$.

This is consistent with (3), and satisfies $\alpha_i \geq \alpha_{i+1}$ for $i = 1, \ldots, n - 1$, and $\sum_{i=1}^n \alpha_i = 1$. The parameter $\beta$ is a measure of consumer attentiveness. Higher values of $\beta$ are associated with higher market shares for the lowest priced store, $\alpha_1$, and lower market shares for the highest priced store, $\alpha_n$. □

Profits for firm $j$ are given by

$$\pi_j(p, \underline{p}, \overline{p}) = R(p_j) s_j(p, \underline{p}, \overline{p}).$$

Each firm $j$ simultaneously chooses a price $p_j \in A_j = \mathbb{R}_+$, with set of pure action profiles $A = \prod_j A_j$. Let $\mathcal{H}^t = A^t$ be the set of $t$-period histories. The set of possible histories is then $\mathcal{H} = \bigcup_{t=0}^\infty \mathcal{H}^t$. A pure strategy for the fringe firm in period $t$ is a mapping from the current history to the set of actions, $\sigma^f_t : \mathcal{H}^t \rightarrow A_f$. A pure strategy for cartel member $j$ is a mapping from the set of all possible histories to the set of actions, $\sigma^j : \mathcal{H} \rightarrow A_j$. Given the strategy profile $\sigma = \{\sigma_j\}_{j=1}^{n-1},\{\sigma^f\}_{t=0}^\infty$, let $a^t_j(\sigma)$ be the period $t$ action for firm $j$ induced by $\sigma$, and let $a^t(\sigma)$ be the associated action profile. Payoffs for cartel member $j$ and fringe firm $f$ are given, respectively, by

$$v^j(\sigma, \underline{p}, \overline{p}) = \sum_{t=0}^\infty \delta^t \pi_j(a^t(\sigma), \underline{p}, \overline{p})$$

and

$$v^f(\sigma, \underline{p}, \overline{p}) = \pi_f(a^t(\sigma), \underline{p}, \overline{p}).$$

For each firm $j$, define the lowest and highest prices observed on the equilibrium path as $p^j = \inf_t a^t_j(\sigma)$ and $\overline{p}_j = \sup_t a^t_j(\sigma)$, respectively. Let $\underline{p}^{(2)}$ and $\overline{p}^{(n-1)}$ be the second lowest minimum and second highest maximum price, respectively. Equilibrium is then formalised with the following definition.

**Definition 1.** (1) A strategy profile $\sigma$ is admissible with respect to $(\underline{p}, \overline{p})$ if $\underline{p} = \underline{p}^{(2)}$ and $\overline{p} = \overline{p}^{(n-1)}$.

(2) A market equilibrium is a triple $(\sigma, \underline{p}, \overline{p})$ such that (i) $\sigma$ is admissible with respect to $(\underline{p}, \overline{p})$, and (ii) $\sigma$ is a subgame perfect Nash equilibrium in the class of admissible strategies.

The admissibility condition imposes consistency between prices and consumer beliefs.\(^7\) The bounds $\underline{p}$ and $\overline{p}$ are determined by the highest and lowest prices charged by at least

\(^7\)Admissibility ensures beliefs are consistent on the equilibrium path, but does not specify beliefs off the equilibrium path. We assume beliefs do not adjust out of equilibrium. de Roos and Smirnov (2013) consider alternative specifications of beliefs in a related context, without qualitatively impacting on the equilibrium outcome.
two firms on the equilibrium path. This requirement rules out pathological cases in which a single firm consistently sets a higher (or lower) price than her competitors and this escapes the attention of consumers. In practice, this eliminates equilibria in which the fringe firm persistently sets a higher price than the cartel. In Example 7, we illustrate the alternative case with $p = \inf_{t, j} a^j_t(\sigma)$ and $\overline{p} = \sup_{t, j} a^j_t(\sigma)$.

To simplify exposition, we define firm strategies in terms of customer-level revenues rather than prices, and we use the shorthand “revenue” to refer to revenue per customer. In addition, we normalise revenues so that $R(\hat{p}) = 1$. This can be achieved through suitable scaling of demand, $D(p)$. This exposition is without loss of generality because: i) by Assumption 1, revenue is strictly increasing in price up to the monopoly price; and ii) firms have no incentive to price above the monopoly price. Let $\overline{r} = R(\overline{p})$ and $\underline{r} = R(p)$ be the revenues associated with the pricing bounds. Given a strategy profile $\sigma$ with symmetric cartel strategies, let $r^s = R(a^j_t(\sigma))$ be the revenue prescribed in period $s$ for each cartel member $j = 1, \ldots, n - 1$. Given cartel revenues $r^s$ and fringe revenue of $r_f$, we use the notation $\pi^s$ to refer to cartel profits in period $s$ as defined in (4). In particular,

$$\pi^s = \pi_j(p, p, \overline{p}), \quad R(p_j) = r^s, \quad j = 1, \ldots, n - 1, \quad R(p_f) = r_f.$$  

### 2.1 Fringe behaviour

Let us first consider the behaviour of the fringe firm. Because the fringe has a myopic perspective, she cannot be disciplined by the cartel. She therefore has a strong temptation to undercut the prices of cartel members. It is only the imperfect attentiveness of consumers that provides a glimmer of hope for the cartel. If consumers are sufficiently inattentive, the cartel will receive residual demand despite fringe undercutting. Further, the fringe firm may resist her desire to undercut if she can still attract a sufficient body of inattentive consumers despite setting a higher price than her rivals.

We consider strongly symmetric equilibria below, so it is sufficient for us to examine reactions as a function of a single cartel revenue. The fringe firm reaction function depends on $r^s$, and the revenue bounds $\underline{r}$ and $\overline{r}$. Let $\hat{r}^s = \max_r r < r^s$. Then, the fringe firm reaction function is given by the function $b$ with

$$b(r^s; \underline{r}, \overline{r}) = \begin{cases} \overline{r} & \text{if } \left( r^s = \underline{r} \text{ and } r^s \leq \alpha_1 \overline{r} \right) \text{ or } \left( r^s \in (\underline{r}, \overline{r}) \text{ and } \alpha_1 r^s \leq \alpha_1 \overline{r} \right); \\ \min \{ 1, \hat{r}^s \} & \text{if } r^s > \overline{r}; \\ \hat{r}^s & \text{otherwise}. \end{cases}$$  

If the cartel price is above the upper price reference bound or below the lower bound, consumer attention is piqued. The fringe therefore prefers to undercut and capture the entire market rather than relent and attract no customers. If instead the cartel price is equal to the

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8In fact, because we consider equilibria with symmetry across cartel members, the lower and upper price bounds will be shared by the $n - 1$ cartel members.

9Here, we run into the “open-set” problem: $\hat{r}^s$ does not exist. As in Tirole (1988), consider the limiting case in which $f$ sets $r^*$, but obtains a higher market share than cartel members, consistent with a lower price.
lower reference bound, undercutting delivers the fringe the entire market, but raising price (relenting) may also be attractive if consumer attentiveness is sufficiently low (i.e. $\alpha_n$ is sufficiently high). For cartel prices between the reference bounds, the payoffs to undercutting depend on $\alpha_1$, while the payoffs to relenting depend on $\alpha_n$. It is then the ratio of $\alpha_n$ and $\alpha_1$ that determines optimal fringe policy.

### 2.2 The cartel problem

To introduce the cartel’s problem, we first define the coordinated cartel strategy $\sigma^k$ as follows.

**Definition 2.** In a $k$-period cycle $\sigma^k$, in period $t$ firm $j$ sets revenue

$$r_j^t = \begin{cases} r(t) & \text{if } t = 0 \text{ or } r_i^\tau = r(\tau) \forall \tau < t, i = 1, \ldots, k-1; \\ 0 & \text{otherwise}, \end{cases} \quad (8)$$

where

$$r(t) = \begin{cases} r^1 & \text{if } t \text{ is divisible by } k; \\ r^s & \text{if } t-s+1 \text{ is divisible by } k, s = 2, \ldots, k, \end{cases} \quad (9)$$

and $r^s \in [0,1], s = 1, \ldots, k$.

The strategy $\sigma^k$ is a grim-trigger strategy that incorporates a time-varying equilibrium path. Equilibrium play dictates that each firm repeats the finite sequence $\{r^1, r^2, \ldots, r^k\}$. Notice that $\sigma^k$ specifies an optimal penal code. If $r > 0$, then there is no profitable deviation in the punishment subgame. Cartel members therefore anticipate no profits following a deviation. A special case of $\sigma^k$ is a constant revenue path in which $r^s = r, s = 1, \ldots, k$, for constant $r$. As a benchmark, the following Lemma examines this case.

**Lemma 1.** There exists no market equilibrium in which cartel members play $\sigma^k$ with $r^s = r \in (0,1]$ for $s = 1, \ldots, k$.

Lemma 1 suggests that a cartel aiming to fix prices at a constant level cannot survive the presence of the fringe. If cartel policies introduce no price variation then the consumer reference bounds are equal, $\underline{r} = \overline{r} = r$. The fringe then has a strong incentive to undercut the cartel price and capture the whole market.

Now, let us consider $\sigma^k$ involving a time varying revenue path. Define $v^s, s = 1, \ldots, k$ to be the continuation value for cartel members starting from period $s$ of the strategy $\sigma^k$:

$$v^1 = \frac{\pi^1 + \delta \pi^2 + \cdots + \delta^{k-1} \pi^k}{1 - \delta^k}, \quad v^2 = \frac{\pi^2 + \delta \pi^3 + \cdots + \delta^{k-1} \pi^1}{1 - \delta^k}, \ldots, \quad v^k = \frac{\pi^k + \delta \pi^1 + \cdots + \delta^{k-1} \pi^{k-1}}{1 - \delta^k},$$

Notice that

$$v^1 = \delta v^2 + \pi^1, \quad v^2 = \delta v^3 + \pi^2, \quad \ldots, \quad v^k = \delta v^1 + \pi^k. \quad (10)$$
Assign $r^k$ to be the lowest point in the firm's revenue sequence:

$$r^k = \min_{s \in \{1, \ldots, k\}} r^s,$$

and let the cartel maximise $v$, defined as follows:

$$v = \max[v^1, \ldots, v^k]. \quad (11)$$

With $v$ as the objective, assigning the minimum point of the cycle to position $k$ in the revenue sequence is without loss of generality. That is, it is equivalent to considering $v^1$ as the cartel objective and allowing the position of the minimum price to be unrestricted.

## 3 Analysis

Equilibrium requires that all firms including cartel members and the fringe competitor have no incentive to deviate. The fringe reaction function (7) stipulates that in each period the fringe firm either marginally undercuts the cartel price or sets the upper price reference bound. The cartel can therefore craft the revenue sequence to manipulate the behaviour of the fringe. If cartel revenue is high, this will induce the fringe to undercut and lower the market share of the cartel. If cartel revenue is low, the fringe relents and the cartel market share is higher. The tradeoff between cartel revenue and market share is then an important consideration for the cartel.

In period $k$, cartel revenue reaches a trough at $r^k$. If the fringe firm undercuts at this time, she captures the whole market and cartel members earn no profits. Therefore, the cartel always induces the fringe firm to relent in period $k$. Below, we argue that the monotonicity of the problem implies that cartel manipulation of the fringe has a cut-off property: if the cartel induces fringe relenting in period $s$ of $\sigma^k$, then it also induces relenting for all periods $\tau > s$. To begin, in Section 3.1, we characterise the cartel problem for situations in which the fringe relents only in period $k$. We then tackle the general problem in Section 3.2.

### 3.1 An undercutting fringe

For the case in which the fringe relents only in period $k$, we set up the cartel’s problem through Lemma 2.

**Lemma 2.** Suppose the fringe undercuts in any period $s$ in which $r^s > r^k$. Then the optimal $k$-period cycle $\sigma^k$ consistent with market equilibrium is given by the solution to the following program:

$$\max_{r^1, \ldots, r^k \in [0,1]} v \quad (12)$$

subject to

$$r^k = \alpha_n \quad \text{and} \quad \bar{r} = 1; \quad (13)$$

$$v^s \geq \alpha_1 r^s, \quad s = 1, \ldots, k-1, \quad (14)$$

$$v^s \geq \alpha_n, \quad s = 1, \ldots, k-1, \quad \text{and} \quad v^k \geq \alpha_{n-1}. \quad (15)$$
The restrictions in equation (13) determine the minimum and maximum revenue on the equilibrium path. At the cycle trough \( r^k \), the cartel must encourage the fringe firm to relent rather than undercut; \( r^k = \alpha_n \) is the highest revenue that achieves this purpose. Optimality of the revenue path ensures that we must have \( \overline{r} = 1 \). Throughout, we impose the equality constraints in (13) and then consider the sustainability of the cartel.

The remaining constraints are sufficient to guarantee internal cartel discipline. For prescribed cartel revenue \( r^s > r^k \), marginally undercutting delivers a market share of \( \alpha_1 \). The constraints in (14) deter this form of deviation. Alternatively, if consumers are sufficiently inattentive, it may be tempting to raise revenue to \( \overline{r} = 1 \) and settle for a lower market share. This deviation is deterred by the constraints in (15). Notice that the fringe firm relents in period \( k \) by setting a revenue of \( r^k = 1 \). A cartel member considering a relenting deviation in period \( k \) will marginally undercut the fringe revenue, obtaining the market share \( \alpha_n - 1 \). In periods \( 1, \ldots, k - 1 \), the fringe firm undercut. In this case, a relenting deviation delivers a market share of \( \alpha_n \). Finally, by undercutting more aggressively below \( r^k \), a firm could capture the whole market and obtain revenue per customer approaching \( r^k \). Notice that with \( r^k = \alpha_n \), this deviation is also prevented if (15) is satisfied.

The cartel’s incentive constraints depend on the market share parameters \( \alpha_1, \alpha_n-1 \), and \( \alpha_n \). In markets populated by attentive consumers, \( \alpha_1 \) may be higher and \( \alpha_n \) and \( \alpha_n-1 \) lower. Consequently, the relative importance of the constraints in (14) and (15) vary with consumer attentiveness. We begin by analysing settings in which prices are highly salient to consumers and the undercutting constraints in (14) are critical. In Section 3.1.2, we consider environments with less attentive consumers; then the relenting constraints of (15) take on increased importance.

### 3.1.1 High salience environments

In this section, we characterise the preliminary problem in which a cartel guards against the undercutting intentions of members without regard to their incentive to relent. Lemma 3 describes the dynamic structure of the solution, while Lemmas 4 to 6 flesh out the details of an explicit solution. We begin by defining the critical discount factor \( \delta_1(\alpha, k, n) \) as the implicit solution to

\[
\delta^k + \frac{(1 - \alpha_n)\alpha_n \delta^{k-1}}{\alpha_1(n-1)} = \left( \frac{\alpha_1 n - 1}{\alpha_1(n-1)} \right)^{k-1}. \tag{16}
\]

**Lemma 3.** Suppose, in a market equilibrium, the fringe undercuts in any period \( s \) in which \( r^s > r^k \). The \( k \)-period cycle \( \sigma^k \) solves the program (12) - (13) if and only if \( \delta \geq \delta_1 \). The cycle is unique and has the following properties:

(i) revenues decline monotonically over the cycle: \( r^1 \geq r^2 \geq \cdots \geq r^k \), with \( r^1 = 1 \) and \( r^k = \alpha_n \);

(ii) \( r^s = \min \left\{ 1, \frac{\delta \alpha_1(n-1)}{\alpha_1(n-1)} r^{s+1} \right\}, s = 1, \ldots, k - 2 \), and \( r^{k-1} = \min \left\{ 1, \frac{(n-1)\delta^2 v^1 + \delta(1-\alpha_n)\alpha_n}{\alpha_1(n-1)} \right\} \);

(iii) \( v^1 \geq v^2 \geq \cdots \geq v^k \) and \( v = v^1 \) is an increasing function of \( \delta \).

Before discussing the Lemma, we offer the following definitions to aid our discussion.
Definition 3. The knot discount factor $\delta_i$ connects two regions: if $\delta < \delta_i$ the equilibrium path has $r^i < 1$; if $\delta \geq \delta_i$ the equilibrium path has $r^i = 1$, for $i = 2, \ldots, k - 1$.

Definition 4. i) An equilibrium price path is a pure sales path if $r^s = 1$ for $s = 1, \ldots, k - 1$. ii) An equilibrium price path is a distinct cycle path if $r^s < 1$ for $s = 2, \ldots, k - 1$.

Recall from the market share function (3), that if consumers are imperfectly attentive, then $\alpha_1 < 1$ and $\alpha_n > 0$: for prices within $[p, \bar{p}]$, a firm who undercuts the price of her competitors does not capture the entire market, and a firm setting a price above her competitors receives positive market share. Equation (16) implies that if $\alpha_1 < 1$ and $\alpha_n > 0$, the critical discount factor $\delta_1 < 1$. Hence, Lemma 3 suggests that, if consumers are imperfectly attentive, an intertemporally dispersed price path may be sustainable for discount factors strictly below one.\textsuperscript{10} This contrasts with our result of Lemma 1 in which we showed that collusion with a fixed price is not sustainable for any discount factor.

The solution has a similar structure to that of de Roos and Smirnov (2013). A set of complementary slackness conditions govern the optimal revenue path. In each period, cartel members either set the monopoly revenue or decrease revenue according to (ii). The knot discount factors $\delta_i$ demarcate paths with different depth and breadth of sales. For $\delta \in [\delta_{k-1}, 1]$, a pure sales path is observed. For $\delta \in [\delta_1, \delta_2)$, we see a distinct cycle path. In between these extremes, cycles involving more extended sales are observed for lower discount factors.

The following three Lemmas complete our characterisation of equilibrium in high salience environments. Lemma 4 allows cross-market comparison. If consumers are less attentive and $\alpha_1$ is lower, deviations involving undercutting provide smaller gains in market share. If $\alpha_n$ is higher, the fringe is more tempted to relent, and the cartel can earn a greater revenue in period $k$. Combining these factors, in markets with less attentive consumers, price dispersion makes collusion sustainable for a greater range of discount factors.

Lemma 4. The critical discount factor $\delta_1(\alpha, k, n)$ is increasing in $\alpha_1$ and decreasing in $\alpha_n$.

Lemma 5 describes the knot discount factors associated with the complementary slackness conditions in part (ii) of Lemma 3.

Lemma 5. The knot discount factors $\delta_i \forall i \in [2, k - 1]$ are determined by the following equations:

$$
\delta^k + \frac{1 - \alpha_1}{\alpha_1(n - 1)} \sum_{j=k-i+1}^{k-1} \delta^j + \frac{\alpha_n(1 - \alpha_n)}{\alpha_1(n - 1)} \delta^{k-i} = \left( \frac{\alpha_1 n - 1}{\alpha_1(n - 1)} \right)^{k-i}. \quad (17)
$$

Part (ii) of Lemma 3 implies that the revenue path is completely determined by $v^1$. We solve explicitly for this in Lemma 6. This allows an analytic solution for the optimal revenue path and accompanying value of the cartel’s objective function.

\textsuperscript{10}Strictly speaking, a cartel unconcerned with relenting needs to satisfy $v^k \geq \alpha_n$ in addition to (14) and (13). This is implied by the constraints in (15) which we consider in Section 3.1.2.
Lemma 6. For $s = 1, \ldots, k - 1$, if $\delta \in [\delta_s, \delta_{s+1})$, then a $k$-period cycle solving the program (12) - (13) yields

$$v^1 = \frac{(1 - \alpha_1) \frac{1 - \delta^s}{1 - \delta} + (1 - \alpha_n) \alpha_n \left( \frac{a_1}{a_1 n - 1} \right)^{k-s-1} \delta^{k-1}}{1 - \left( \frac{a_1}{a_1 n - 1} \right)^{k-s-1} \delta^k (n-1)}.$$ (18)

Lemmas 3 - 6 describe the solution to the cartel’s optimal price setting problem in high salience environments. We illustrate this case with Example 2.

Example 2. Consider the following setting. There are $n = 4$ firms consisting of three cartel members and a fringe. Cartel members choose strategies with cycle length $k = 4$. The lowest priced firm obtains a market share $\alpha_1 = 0.35$, and the highest priced firm obtains share $\alpha_4 = 0.2$. With these parameter values, the fringe relents only in period $k$ and the relenting constraints in (15) do not bind for the cartel. Figure 1 displays the resulting equilibrium revenue path for a range of discount factors. The vertical axis shows revenue and the horizontal axis indexes discount factors. Reading the figure vertically reveals revenue in each period of a cycle for a given discount factor.

Sustainability of collusion depends on relative not absolute prices, so there is no benefit to lowering $r^1$ below the monopoly level. To induce the fringe to relent in period $k = 4$ requires $r^k \leq \alpha_n$. The remaining revenues are determined by the discount factor. For sufficiently high $\delta$, monopoly pricing can be sustained in periods 2 and 3. For $\delta$ below the knot discount factor $\delta_3$, $r^3$ must be lowered in order to satisfy the undercutting constraints in (14). Similarly, for $\delta < \delta_2$, $r^2$ must be lowered to satisfy these constraints.

3.1.2 General salience environments

We now consider the impact of the relenting constraints on the optimal revenue path of the cartel. In Lemma 7, we describe the set of parameters for which the relenting constraints impinge on cartel policies.

We begin with two definitions. Given the revenue sequence $\{r^1_s\}_{s=1}^{k}$, define the discount factor $\hat{\delta}(\alpha, k, n)$ implicitly by

$$v^k(\hat{\delta}) = \alpha_{n-1},$$ (19)

where $v^k$ is determined by (18) and (10). When the cartel’s most pressing concern is the incentive of members to relent, $\hat{\delta}$ will be the critical discount factor. Next, define the following function:

$$g(\alpha, k, n) = \frac{\alpha_{n-1}}{\alpha_1} - \left( \frac{\alpha_1 n - 1}{(n-1)\alpha_{n-1} - \alpha_n(1 - \alpha_n)} \right)^{k-1}.$$ (20)

The function $g(\alpha, k, n)$ demarcates regions of the parameter space for which the cartel needs greater internal vigilance towards undercutting or relenting, as described in Lemma 7.
Lemma 7. Suppose the revenues \( \{r_s^k\}_{s=1}^k \) solve the program (12) - (13), and let the associated continuation values be \( \{v^s(\delta)\}_{s=1}^k \). Consequently,

i) if \( g(\alpha, k, n) \leq 0 \), then the revenues \( \{r_s^k\}_{s=1}^k \) solve the program (12) - (15);

ii) if \( g(\alpha, k, n) > 0 \), then there exists a critical discount factor \( \delta^* \), implicitly defined by (19), such that the revenues \( \{r_s^k\}_{s=1}^k \) solve the program (12) - (15) if \( \delta \geq \delta^* \), and there is no solution to the program (12) - (15) if \( \delta < \delta^* \).

Lemma 7 determines whether internal defense against undercutting or relenting is the more pressing concern of the cartel. If consumers are attentive, \( \alpha_1 \) will take on higher values and \( \alpha_n \) lower values. The function \( g(\alpha, k, n) \) will be negative in these circumstances and the undercutting constraints (14) determine cartel feasibility. If consumers are inattentive, relenting becomes more attractive relative to undercutting and the constraints in (15) determine feasibility.

Lemma 7 also highlights the contrasting roles of the undercutting and relenting constraints. The undercutting constraints determine the shape of the optimal revenue path if collusion is feasible. The relenting constraints play a binary role: they influence cartel feasibility, but they have no impact on the shape of the optimal revenue path. The reason is intuitive. To relax the relenting constraints, cartel revenues must be raised to increase the continuation value \( v^k \). However, an optimising cartel will already be maximising cartel revenues subject to the undercutting constraints (14).

In light of both undercutting and relenting constraints, the cartel's critical discount factor...
is given by $\delta^* = \max \{\delta_1 \hat{\delta}\}$. In particular,

$$\delta^*(\alpha, k, n) = \begin{cases} 
\delta_1(\alpha, k, n) & \text{if } g(\alpha, k, n) \leq 0; \\
\hat{\delta}(\alpha, k, n) & \text{if } g(\alpha, k, n) > 0.
\end{cases} \quad (21)$$

If the fringe undercuts in any period $s$ in which $r^s > r^k$, then collusion is sustainable for $\delta \geq \delta^*(\alpha, k, n)$.

### 3.2 Manipulation of the fringe

We now consider the cartel's general problem. In addition to the internal incentive constraints of cartel members, the cartel also manipulates the behaviour of the fringe. As a stepping stone, we first consider cartel behaviour in period $k - 1$. We illustrate with Example 3, in which the cartel induces the fringe to relent in period $k - 1$. Through Lemma 8, we then provide sufficient conditions for equilibria with this property. Examples 4 and 5 provide intuition for this result. We then consider the cartel's general problem. Proposition 1 characterises the optimal revenue path adopted by a cartel concerned with both internal incentive constraints and opportunities for fringe manipulation.

Consider the cartel's incentives in period $k - 1$. Given cartel revenue $r^{k-1}$, the fringe prefers to relent in period $k - 1$ if $\alpha_n \geq \alpha_1 r^{k-1}$. The cartel can induce the fringe to relent by setting $r^{k-1} = \alpha_n / \alpha_1$, yielding cartel member profits of $\pi^{k-1} = \frac{1-\alpha_n}{n-1} \frac{\alpha_n}{\alpha_1}$. If the fringe instead undercuts, cartel members earn $\pi^{k-1} = \frac{1-\alpha_1}{n-1} r^{k-1}$. There is therefore a threshold revenue level of

$$r^{k-1} = \frac{\alpha_n (1-\alpha_n)}{\alpha_1 (1-\alpha_1)}, \quad (22)$$

below which the cartel prefers to encourage the fringe to relent. Example 3 illustrates.

**Example 3.** Consider the following minor variation to Example 2. There are $n = 4$ firms consisting of three cartel members and a competitive fringe. Cartel members choose strategies with cycle length $k = 4$. The lowest priced firm obtains a market share $\alpha_1 = 0.34$, and the highest priced firm obtains share $\alpha_4 = 0.2$. Figure 2 displays the resulting equilibrium revenue path for a range of discount factors. The vertical axis shows revenue and the horizontal axis indexes discount factors. Reading the figure vertically reveals revenue in each period of a cycle for a given discount factor.

The only change from Example 2 is a decrease in $\alpha_1$ from 0.35 to 0.34. For discount factors above $\delta'$, there is no qualitative difference to our earlier example. For discount factors below $\delta'$, the optimal cartel revenue $r^3$ crosses the cut-off level $\alpha_n (1-\alpha_n) / (\alpha_1 (1-\alpha_1))$. The cartel then induces the fringe to relent by setting a revenue of $\alpha_n / \alpha_1$. Consequently, the fringe relents in periods $k$ and $k - 1$ for discount factors between $\delta_1$ and $\delta'$.

In Example 3, the cartel induces the fringe to relent before the last period for a range of discount factors, $\delta \in [\delta_1, \delta')$. In Lemma 8, we implicitly define sufficient conditions on the remaining parameters $(\alpha, k, n)$ for the fringe to undercut in period $k - 1$. 

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Figure 2: Optimal cartel revenues with fringe manipulation

Lemma 8. If $\delta \geq \delta^*(\alpha, k, n)$, then the fringe undercuts the cartel in every period $s \leq k - 1$ if

$$v^k \geq \frac{\alpha_n (1 - \alpha_n) \alpha_1 n - 1}{\alpha_1 (1 - \alpha_1) \delta^*(n - 1)}, \quad \text{where} \quad v^k = \max \left\{ \alpha_{n-1}, \frac{\alpha_n (1 - \alpha_n)}{n-1} + \delta_1 \alpha_1 \right\}. \quad (23)$$

The right hand side of equation (23) describes the cartel value in period $k$ when $\delta = \delta^*$ if the optimal revenue path is chosen and the fringe undercuts in every period $s \leq k - 1$. The value of $v^k$ depends on whether the undercutting or relenting constraints are binding. This implicitly defines $v^k$ as a function of the parameter vector $(\alpha, n, k)$. The left hand side of (23) provides the conditions required for undercutting. If $v^k$ is sufficiently high when the fringe undercuts, then the cartel is happy to induce undercutting by the fringe. At $\delta = \delta^*$, the conditions are necessary and sufficient. For $\delta > \delta^*$, the conditions are sufficient.

We use Examples 4 and 5 to provide intuition for Lemma 8.

Example 4. There are $n = 4$ firms consisting of three cartel members and a fringe. Cartel members choose strategies with cycle length $k = 4$. In Figure 3, we plot iso-revenue curves, described below, for a range of market share parameter vectors. The horizontal axis indexes the share of the lowest-priced firm, $\alpha_1$. The vertical axis depicts the share of the highest-priced firm, $\alpha_4$. For our simulations, we set $\alpha_3 = \alpha_4$. Note also that $\alpha_2$ does not influence the equilibrium revenue path.

The blue line, labeled $\delta_1 = \hat{\delta}$, describes combinations of $\alpha_1$ and $\alpha_4$ for which undercutting and relenting are equally tempting. The red and green lines are iso-revenue lines, describing parameter combinations for which the cartel optimally sets the same revenue in period $s = 3$. If $\alpha_1$ is large and $\alpha_n$ is small, undercutting presents a more attractive deviation for the cartel than relenting. In this case, the undercutting (red) iso-revenue line, labeled
Consider the undercutting iso-revenue line. For \( \delta = \hat{\delta} \), this line describes combinations of \( \alpha_1 \) and \( \alpha_4 \) for which the cartel is indifferent between inducing the fringe to relent or undercut in period \( s = 3 \). Loosely speaking, for higher values of \( \alpha_1 \) and lower values of \( \alpha_4 \), the fringe is more tempted to undercut, suggesting a positive relationship. Complicating the analysis, the market share parameters also determine the critical discount factor \( \delta_1 \) (recall Lemma 4). Moving away from the iso-revenue line, lower values of \( \alpha_1 \) are associated with a lower critical discount factor \( \delta_1 \) and a lower value of \( r_3 \). This encourages the fringe to relent. Thus, to the left of the undercutting iso-revenue line, the fringe would relent in period \( s = 3 \) if undercutting were the cartel’s principal concern.

Next, consider the relenting iso-revenue line. For \( \hat{\delta} = \hat{\delta} \), this line describes combinations of \( \alpha_1 \) and \( \alpha_4 \) for which the cartel is indifferent between inducing the fringe to relent or undercut in period \( s = 3 \). The slope of the relenting line differs from the undercutting line because \( \hat{\delta} \) adjusts differently to \( \delta_1 \) in response to changes in the market share parameters. As we depart from the iso-revenue line, higher values of \( \alpha_1 \) relax the cartel’s relenting constraints, leading to lower values of \( \hat{\delta} \), and lower values of \( r_3 \). This again encourages the fringe to relent. Hence, to the right of the relenting iso-revenue line, the fringe relents in period \( s = 3 \).

Combining this information, when \( \delta = \delta^* \), the fringe relents in the region between the two iso-revenue lines to the right of the intersection of the two lines. Finally, the two black lines indicate feasible combinations of market share parameters. Recall that we have imposed \( \alpha_3 = \alpha_4 \). The two lines correspond to the extreme cases where \( \alpha_2 = \alpha_1 \) and \( \alpha_2 = \alpha_4 \). The area between these two lines further demarcates the range of parameters for which the fringe relents in period \( s = 3 \).

**Example 5.** As in Example 4, there are \( n = 4 \) firms comprising three cartel members and a competitive fringe. Cartel members choose strategies with cycle length \( k = 4 \). The one change we make from the previous example is to set \( \alpha_3 = 1.05\alpha_4 \). We plot iso-revenue lines in the same manner, and illustrate in Figure 4. For \( \delta = \delta^* \), with these parameter values, there is no feasible combination of market share parameters \( \alpha_1 \) and \( \alpha_4 \) for which the fringe is induced to relent in period \( s = 3 \).

**The optimal revenue path of the cartel**

Below, we show that the fringe manipulation problem reduces to a simple cut-off rule: the cartel decides in which period the fringe should begin relenting. To set up the problem, we first construct a sequence of \( k - 1 \) revenue paths, corresponding to alternative cut-off policies.

Define the sequence of cartel revenue vectors and associated cartel value \( \{ r^{s,(i)}, v^{1,(i)} \}_{s=1}^k \),
Figure 3: Optimal fringe manipulation at $\delta = \delta^*$

Figure 4: At $\delta = \delta^*$ the cartel prefers the fringe to undercut
for \(i = 1, \ldots, k-1\) as follows. Let \(\{r_{s(i)}\}^k_{s=1}\) solve the program (12) - (13). For \(i = 2, \ldots, k-1\), let

\[
r_{s(i)} = \begin{cases} 
\frac{\alpha_n}{\alpha_n} & \text{if } s = k, \\
\delta^{i+2}\frac{\alpha_1}{\alpha_1(1-\alpha_n/a_n)^{n-1}} & \text{if } s = k - i, \ldots, k-1, \\
\min\left\{1, \frac{\alpha_n}{\alpha_n(1-\alpha_n/a_n)^{n-1}} r_{s+1(i)}\right\} & \text{if } s = 2, \ldots, k-i-1, \\
1 & \text{if } s = 1.
\end{cases}
\]

(24)

The path \(\{r_{s(i)}\}^k_{s=1}\) corresponds to the optimal cartel revenue path when the fringe relents only in the last period. This is the solution we described in Section 3.1. The path \(\{r_{s(i)}\}^k_{s=1}\) describes the optimal revenue path when the fringe relents in the last \(i\) periods of each sequence.

Because cartel incentives depend only on relative prices, the cartel will always optimally set a revenue of 1 in period 1. In Proposition 1, we show that the cartel’s optimal revenue path is monotonically decreasing. Therefore the period \(k\) revenue \(r_k = r\). It follows that the cartel wishes to induce the fringe to relent in period \(k\). Otherwise, the fringe will capture the entire market. The fringe is indifferent between relenting and undercutting at a revenue of \(\alpha_n\). This constrains \(r_k = \alpha_n\). If in period \(s\), the cartel wishes to induce relenting by the fringe, it will set \(r_s\) as high as possible while providing the incentive to relent. For \(s < k\), the fringe firm will obtain a market share of \(\alpha_1\) by marginally undercutting and a market share of \(\alpha_n\) by relenting. The fringe is therefore indifferent between relenting and undercutting if \(r_s = \alpha_n/\alpha_1\). In the remaining periods, \(s = 2, \ldots, k-i\), optimal revenues are determined by complementary slackness conditions that resemble the solution of Lemma 3.

The fringe is manipulated by the cartel cut-off policy. Define the sequence of fringe revenue vectors \(\{r_{s(i)}\}^k_{i=1}\), \(i = 1, \ldots, k-1\) as follows.

\[
r_{s(i)} = \begin{cases} 
\hat{r}_{s(i)} & \text{if } s \leq k-i \text{ and } a < r_{k-i(i)}, \\
1 & \text{if } s > k-i.
\end{cases}
\]

(25)

We can now solve for the optimal cartel policy. Proposition 1 provides the conditions required for equilibrium, introduces the cut-off rule used to evaluate whether the cartel encourages the fringe to undercut or relent, and describes the cartel’s optimal revenue path.

**Proposition 1.** Label \(\{r_s, r_f\}^k_{s=1}\) as a candidate solution and assign \(\{r_s, r_f\}^k_{s=1}\) as follows:

\[
\begin{align*}
\text{if } & r_{k-1(i)} > a, \text{ then } \{r_s, r_f\}^k_{s=1} = \{r_{s(i)}^{(1)}, r_f\}^k_{s=1}, \\
\text{if } & r_{k-i+1(i)} < a < r_{k-i(i)}, \text{ then } \{r_s, r_f\}^k_{s=1} = \{r_{s(i)}^{(1)}, r_f\}^k_{s=1}, \quad i = 2, \ldots, k-1,
\end{align*}
\]

(26)

where \(r_{s(i)}^{(1)}, i = 2, \ldots, k-1\) and \(a\) are defined in (22), \(r_{s(i)}^{(1)}\) is defined in Lemma 3, and \(r_{f(i)}^{(1)}, i = 1, \ldots, k-1\) are defined in (25).
Let \( \{v^s\}_{s=1}^k \) be the associated cartel continuation values. If \( v^1 \geq \alpha_1 \) and \( v^k \geq \alpha_{n-1} \), then \( \{r^s, r^s_f\}_{s=1}^k \) is the optimal market equilibrium for the cartel. Otherwise, there is no market equilibrium with positive value for the cartel.

Proposition 1 describes an inductive structure. If the fringe relents only in the last period, then \( \{r^s,(1)\}_{s=1}^k \) is the optimal revenue path described in Section 3.1. If \( r^{k-1,(1)} > a \), then the cartel prefers the fringe to undercut in period \( k - 1 \), and this is indeed the optimal revenue path. Alternatively, if the cartel prefers the fringe to relent, then we consider the path \( \{r^s,(2)\}_{s=1}^k \). If \( r^{k-2,(2)} > a \), then the fringe undercuts in period \( k - 2 \), and this revenue path is optimal. We apply this cut-off rule recursively to obtain the optimal revenue path.

The shape of the optimal revenue path is determined by the cartel’s undercutting constraints. The sustainability of the optimal revenue path depends on two final conditions. The constraint \( v^1 \geq \alpha_1 \) is the analogue of the critical undercutting discount factor, \( \delta_1 \). Constraints 2 to \( k - 1 \) in (14) can be negotiated by adjusting relative prices. If the undercutting constraint in period 1 is not satisfied, relative prices cannot be further adjusted and the cartel is unsustainable. The constraint \( v^k \geq \alpha_{n-1} \) corresponds to the critical relenting discount factor, \( \hat{\delta} \). Because \( v^s \) is monotonically decreasing in \( s \), this is a sufficient condition to ensure cartel members have no incentive to relent.

4 Discussion

In this section, we present examples illustrating adjustment of the market share function (Section 4.1), the admissibility condition (Section 4.2) and the number of fringe firms (Section 4.3).

4.1 Market shares

The specification of the market share function (3) aids the analytic solution to the cartel’s optimal dynamic price setting problem. In this section, we argue the robustness of our solution to more general specifications. As de Roos and Smirnov (2013) note in a related context, the optimal revenue path is determined by discounting and deviation payoffs rather than the shape of the market share function per se. We use Example 6 as a vehicle for additional discussion.

Example 6. Consider the following setting. There are \( n = 4 \) firms, including 3 cartel members and a fringe firm. Cartel members choose strategies with cycle length \( k = 4 \). If all prices are in the interval \([p, \bar{p}]\), then market shares are determined by (3), with \( \alpha_1 = 0.5 \) and \( \alpha_4 = 0.05 \) and \( \alpha_2 \) and \( \alpha_3 \) unrestricted. In the event that at least one firm sets a price below \( p \), market shares are determined by the vector \( \alpha' \), with \( \alpha'_1 = 0.8 \) and \( \alpha'_3 = 0.01 \), and \( \alpha'_2 \) and \( \alpha'_4 \) unrestricted. That is, in the event that a firm sets an unusually low price, she attracts the attention of more consumers, but not all consumers as in (3). This adjustment to the market share function has two immediate implications. First, even if the fringe undercuts at the cycle trough \( p \),
cartel members retain a positive market share. Second, a zero-value penal code is no longer available. Instead, an optimal penal code must be constructed.\footnote{In the optimal penal code, a deviating firm can be held to her security level in the continuation game, with value $\alpha_3'$. See de Roos and Smirnov (2013) for the construction of the optimal penal code.}

Figure 5 shows the resulting optimal cartel revenue path. The horizontal axis indexes alternative values of the discount factor, and the vertical axis shows revenues for each period of the cycle. There are two qualitative differences relative to Example 2. First, if cartel members are sufficiently patient, collusion with a fixed price can now be supported. This follows because the fringe no longer captures the entire market when she undercuts the cartel price.\footnote{In this equilibrium, and others discussed below, the fringe persistently undercuts the lower bound price $\bar{p}$. This suggests that $\bar{p}$ may be undefined. We address this issue in the same manner as the fringe reaction function (7): the fringe undercuts by setting a price equal to $\bar{p}$ and obtains the market share $\alpha_1'$.}

For lower values of the discount factor, a fixed price is no longer viable. Price dispersion is required to obfuscate consumer price comparison. Second, the cartel induces the fringe to undercut in all periods of a price cycle. This also stems from the inability of the fringe to attract all consumers by undercutting at the cycle trough. For values of $r^4$ below the critical level $\alpha_4(1 - \alpha_4)/(\alpha_1'(1 - \alpha_1'))$, the cartel would instead elect to encourage fringe relenting. In our example, this situation does not arise. In all other respects, the cartel’s optimal revenue path resembles our earlier examples.

\[ \frac{\alpha_4(1 - \alpha_4)}{(1 - \alpha_1')} \]

\[ \frac{\alpha_4(1 - \alpha_4)}{(1 - \alpha_1')} \]
4.2 The admissibility condition

According to Definition 1, a strategy profile is admissible if the lower and upper bounds of the reference price range are determined by the prices of the firms with the second lowest and second highest prices, respectively. This rules out equilibria in which a single firm persistently prices above or below the price of her competitors and this escapes the attention of consumers. In this section, we employ Example 7 to illustrate an alternative definition in which the reference bounds are determined by the lowest and highest prices among all firms.

Example 7. There are 5 firms, including 4 cartel members and 1 fringe firm. Cartel members choose cycles of length \( k = 3 \), and market share parameters are given by \( a_1 = 0.36 \), \( a_n = a_{n-1} = 0.16 \). We modify Definition 1 so that the lower and upper pricing bounds are determined by \( \underline{p} = \min_{t, j} a_j^t(\sigma) \) and \( \overline{p} = \max_{t, j} a_j^t(\sigma) \). Figure 7 illustrates the optimal cartel price path for a range of discount factors. For \( \delta \geq \delta_4 \), the solution has the same character as in our main model.

Consider first the range \( \delta \in (\delta_4, \delta_5) \). In this region, undercutting constraint 2 is binding and lower values of \( \delta \) compel the cartel to reduce \( r^2 \) in order to adjust relative prices. For \( \delta < \delta_4 \), undercutting constraint 1 is also binding. Under Definition 1, there is no further avenue to adjust relative prices because any decrease in \( r^1 \) translates directly into a decrease in \( \overline{p} \). As a result, collusion is not sustainable for \( \delta < \delta_4 \). However, with our modification, \( \overline{p} \) can be determined by the unilateral action of the fringe. This permits the cartel to lower \( r^1 \) without impacting on \( \overline{p} \), introducing an avenue for further adjustment in relative prices. The additional knot discount factors \( \delta_2 \) and \( \delta_3 \) represent thresholds for fringe manipulation. For \( \delta < \delta_3 \), the fringe relents in period 2 and 3, and for \( \delta < \delta_2 \), the fringe sets the monopoly price in every period.

4.3 Multiple fringe firms

So far we have considered a cartel facing a single strategic fringe firm. In this section, we use an example to suggest that our principal argument generalises to more than one fringe competitor. In Example 8, we show that a minor modification to the definition of market equilibrium gives rise to a unique profitable equilibrium for the cartel involving a pure sales path.

Example 8. There are \( n \) firms consisting of \( n-2 \) cartel members and two fringe firms. Suppose that \( a_{n-1} = a_n \) and cartel members choose strategies with cycle length \( k \). Under Definition 1, the price reference bounds are determined by the highest and lowest prices set by at least two firms on the equilibrium path. As in Example 7, we modify Definition 1 for this example so that a single firm is sufficient to determine the price reference bounds.

Equilibrium imposes two requirements on fringe behaviour: in each period, there must be no incentive to adjust revenue; and each fringe firm must earn the same profits as her competitor. In equilibrium each fringe firm sets a different price; otherwise, there would be an incentive to marginally undercut the price of one’s fringe competitor. Given cartel
revenue $r^s$ and the market share function (3), this means that the fringe revenues must be 1 and $\dot{r}^s$. In each period $s$ with $r^s > r^k$, equal fringe profits then requires $\alpha_n = \alpha_1 r^s$. In period $k$, we must have $\alpha_n = r^k$. These conditions pin down the revenue path with $r^k = \alpha_n$ and $r^s = \alpha_n / \alpha_1$ for $s = 1, \ldots, k-1$. This revenue path is sustainable for $\delta \geq \delta^*$ with $\delta^* < 1$. Figure 7 illustrates equilibrium cartel and fringe pricing over time for the case $k = 4$, $\alpha_1 = 0.35$, and $\alpha_{n-1} = \alpha_n = 0.2$.

5 Conclusion

In this paper, we examined the problem faced by a price-setting cartel negotiating a fringe competitor in the presence of imperfectly attentive consumers. Because each firm produces an identical product, the cartel is vulnerable to price undercutting by the fringe. Price dispersion then becomes a critical weapon for the cartel. With fixed prices, undercutting by the fringe captures the attention of consumers, and the fringe is able to destroy the cartel. A dispersed price path hampers consumer price comparison and mitigates the market share gains of an aggressive fringe, permitting the survival of the cartel.

We characterise the optimal revenue path of the cartel in the class of infinitely repeated finite-length cycles. The cartel must navigate both the internal incentive constraints of members and the impulses of the fringe. The optimal path involves a sequence of monopoly prices, punctuated with sales. Longer and deeper sales are required to satisfy internal incentive constraints if cartel members are less patient. The cartel manipulates the fringe through the use of a simple cut-off rule. The cartel allows the fringe to undercut the price of the cartel
Figure 7: Cartel revenues with two fringe firms

in the early stages of a cycle. Beyond a cut-off date in each cycle, the fringe is coaxed to relent and raise prices above the cartel.

Appendices

A Proofs

Proof of Lemma 1

Proof. Suppose there is a market equilibrium in which the cartel plays $\sigma^k$ with $r^s = r \in (0, 1], s = 1, \ldots, k$. By Definition 1, $r = r = r$. Let the fringe revenue be $r_f$. Suppose $r_f \geq r$. Because $r = r$, the fringe attracts no customer. The fringe firm could capture the entire market by marginally undercutting the cartel price. This is a profitable deviation, leading to a contradiction. Suppose instead that $r_f < r$. Then the fringe firm captures the whole market and cartel members earn no profits. Each cartel member has an incentive to undercut the price of the fringe firm, which is again a contradiction. 

Proof of Lemma 2

Proof. 1. Suppose $r^k > 0$. Infinite reversion to marginal cost pricing is then an optimal penal code. To see this, note that $r^k = r > 0$. Now consider the stage game of our model consisting of a single period of play. Assumption 2 implies that there exists a Nash equilibrium to the stage game in which each firm sets a price of zero. This is because, with $p > 0$, any firm $j$ setting price $p_j > 0$ will obtain market share of zero if her competitors set a price of zero. Therefore, there is a Nash equilibrium to the stage game in which firms price at marginal
cost and obtain zero profits, and infinite reversion to marginal cost pricing is an optimal penal code.

2. A standard application of the one-shot deviation principle is possible. See, for example, Fudenberg and Tirole (1991). We restrict attention to one-shot deviations below.

3. Next note that in an optimal cartel solution, the fringe firm must relent in period $k$ and in any period in which $r^s = r^k$. Suppose otherwise. Then, the fringe firm will capture the whole market and $\pi^s = 0$, which is not optimal for the cartel. To induce relenting by the fringe we therefore require

$$r^k \leq \alpha_n \bar{r}. \quad (27)$$

4. If a cartel member were to deviate by choosing a revenue below $r^k$, she could obtain a market share of 1. To prevent this deviation requires

$$v^s \geq r^k, \quad s = 1, \ldots, k. \quad (28)$$

5. If a cartel member wishes to deviate by choosing a revenue $r \in [r^k, r^s)$, it is optimal to minimally undercut. If $r^s > r^k$, then prevention of this deviation requires

$$v^s \geq \alpha_1 r^s, \quad s = 1, \ldots, k. \quad (29)$$

If $r^s = r^k$, then the constraints in (28) will prevent marginal undercutting.

6. An alternative deviation for cartel members is to raise revenue above $r^s$. The optimal such deviation involves choosing revenue $\bar{r}$. To prevent this deviation requires

$$v^s \geq \alpha_n \bar{r}, \quad s = 1, \ldots, k. \quad (30)$$

7. From (27) $r^k \leq \alpha_n \bar{r}$. Next, we show that

$$r^k = \alpha_n \bar{r}. \quad (31)$$

Suppose otherwise that $r^k < \alpha_n \bar{r}$. Then, by raising $r^k$, $v^s$ could be increased for $s = 1, \ldots, k$ without violating any constraints, leading to a contradiction.

8. A similar argument implies $r^k > 0$, confirming our supposition in 1. Suppose otherwise that $r^k = 0$. Then $v^s$ could be increased and the value of punishment decreased by raising $r^k$, establishing a contradiction.

9. All of our constraints are homogeneous of degree one with respect to revenues. We show by contradiction that

$$\bar{r} = \max_s r^s = 1. \quad (32)$$

By our normalisation of revenues, $r^s \leq 1 \forall s$. Suppose that $\max_s r^s < 1$. Introduce the variables $\tilde{r}^s = r^s / (\max_i r^i) \forall s$. Because of first degree homogeneity, our transformed variables must satisfy the constraints. Given that $\tilde{r}^s > r^s \forall s$, it means that $v^s$ has been increased $\forall s$. Consequently, there is a contradiction and $\max_i r^i = 1$.

10. Combining equations (29), (30), (31) and (32) leads to the constraints in the Lemma. \qed
Proof of Lemma 3

Proof. 1. Number the constraints in (14) 1, ..., k − 1, where constraint s involves \( v^s \).
2. Let us assume that \( \delta > \frac{\alpha_1 n - 1}{a_1(n-1)} \). We show later that this condition is satisfied for \( \delta \geq \delta_1 \).
3. Show that for each \( s = 1, ..., k \), constraint s is either binding or \( r^s = 1 \). Consider constraint s and suppose otherwise that the constraint is not binding and that \( r^s < 1 \). By increasing \( r^s \) until either \( r^s = 1 \) or constraint s is binding, the objective function \( v \) is increased and all the constraints in (14) are satisfied, leading to a contradiction.
4. Next, we show that for each \( s = 1, ..., k - 1 \), if \( r^s = 1 \) then \( r^{s-1} = 1 \). If \( r^s = 1 \) then constraint s becomes \( v^s \geq \alpha_1 \). Constraint s − 1 stipulates \( v^{s-1} \geq \alpha_1 r^{s-1} \). From (10), \( v^{s-1} = \delta v^s + \pi^{s-1} \).
Because the fringe undercuts, \( \pi^{s-1} = \frac{1 - \alpha_1}{a_1(n-1)}r^{s-1} \). Combining this information into constraint \( s - 1 \) gives \( v^s \geq \frac{\alpha_1 n - 1}{a_1(n-1)} r^{s-1} \). With \( \delta > \frac{\alpha_1 n - 1}{a_1(n-1)} \), this implies that \( r^{s-1} = 1 \) satisfies constraint \( s - 1 \).
5. We next show that \( r^s = \min \{ 1, \delta \frac{\alpha_1 (n-1)}{a_1} r^{s+1} \}, s = 1, ..., k - 2 \). First, note that if constraint \( s + 1 \) is not binding then \( r^s = 1 \). Alternatively, suppose constraint \( s + 1 \) is binding. We can use (10) to transform constraint s to obtain \( \delta v^{s+1} + \frac{1 - \alpha_1}{n-1} r^s \geq \alpha_1 r^s \). Using the fact that constraint \( s + 1 \) is binding, we obtain \( \delta \alpha_1 (n-1) r^{s+1} + (1 - \alpha_1) r^s \geq \alpha_1 (n-1) r^s \) or \( r^s \leq \frac{\delta \alpha_1 (n-1) r^{s+1}}{\alpha_1} \). Applying point 4. above yields our desired result.
6. Let us prove that \( r^{k-1} = \min \{ 1, \frac{(n-1)\delta^2 v^1 + \delta (1 - a_n) a_n}{\alpha_1(n-1)} \} \). From (10), \( v^{k-1} = \delta v^k + \pi^{k-1} \). Because the fringe undercuts, \( \pi^{k-1} = \frac{1 - \alpha_1}{n-1} r^{k-1} \). Integrating the above into constraint \( k - 1 \), we obtain \( \delta v^k + \frac{1 - \alpha_1}{n-1} r^{k-1} \geq \alpha_1 r^{k-1} \). Using the fact that \( r^k = \alpha_n \), and noting that the fringe relents in period k, we obtain \( v^k = \delta v^1 + \frac{1 - \alpha_n}{n-1} a_n \). Combining these relationships gives \( r^{k-1} \leq \frac{(n-1)\delta^2 v^1 + \delta (1 - a_n) a_n}{\alpha_1(n-1)} \). Applying point 4. above yields our desired result.
7. Let us prove that \( v = v^1 \). Note that point 5. and the fact that \( \delta > \frac{\alpha_1 n - 1}{a_1(n-1)} \) imply \( r^1 \geq r^2 \geq \cdots \geq r^{k-1} \). In addition, note that if \( r^s \leq \alpha_n / \alpha_1 \) for \( s \leq k - 1 \), then it is optimal for the fringe to relent in period s. Given that we consider the scenario when fringe relents only in period k, that means \( r^s > \alpha_n / \alpha_1 \) and consequently \( r^{k-1} > r^k \). The result follows immediately.
8. Next, we show that \( \forall \delta \), the optimal sequence \( \{ r^i \}_{i=1}^{k-1} \) if it exists, must be unique. Suppose instead that there are two sequences. \( v^1 \) must be the same for both sequences, otherwise the one with the higher \( v^1 \) is chosen. This will uniquely determine the values of \( r^k \) and \( r^{k-1} \) and then, recursively, \( r^{k-2}, \ldots, r^1 \). The optimal sequence is therefore unique.
9. Next, we derive \( \delta_1 \). Consider the situation when all constraints are binding. From constraint \( k - 1 \) it follows that \( v^{k-1} = \alpha_1 r^{k-1} \). Using (10) and \( \pi^{k-1} = \frac{1 - \alpha_1}{n-1} r^{k-1} \), it follows that \( v^k = \frac{a_1 n - 1}{\delta \alpha_1(n-1)} v^{k-1} \). Continuing this process results in

\[ v^k = \left( \frac{\alpha_1 n - 1}{\delta \alpha_1(n-1)} \right)^{k-i} v^i, \quad i = 1, \ldots, k - 1. \] (33)

In particular, \( v^k = \left( \frac{\alpha_1 n - 1}{\delta \alpha_1(n-1)} \right)^{k-1} v^1 \). Then, using \( v^1 = \alpha_1 \), \( v^k = \delta v^1 + \pi^k \), \( \pi^k = \frac{1 - \alpha_n}{n-1} r^k \), and \( r^k = \alpha_n \), results in equation (16).
10. Let us prove that \( v^1 > v^2 > \cdots > v^k \). Without loss of generality, suppose that \( r^{s-1} = 1 \) and \( r^s < 1 \). From equation (33) it immediately follows that \( v^s > v^{s+1} > \cdots > v^k \). Next, note
that for \( i < s \), \( \pi^i = \frac{1-a_i}{n-1} \) and for \( i \geq s \), \( \pi^i = \frac{1-a_i}{n-1} \). Therefore, \( (1-\delta)v^i < \frac{1-a_i}{n-1} \) for \( i < s \). From equation (10), \( v^i = \delta v^{i+1} + \frac{1-a_i}{n-1} \). Combining these two expressions gives \( v^i - \delta v^{i+1} > (1-\delta)v^i \) or \( v^i > v^{i+1} \) for \( i < s \).

11. Next, note that \( v^1(\delta) \) is strictly increasing in \( \delta \), where \( v^1(\delta) \) is the value of \( v^1 \) associated with the optimal revenue sequence for discount factor \( \delta \). Consider any two discount factors \( \delta \) and \( \delta' \) with \( \delta' > \delta \). Let \( r = \{r^1_k\}_{k=1}^n \) be the optimal revenue sequence associated with \( \delta \) and \( r' \) be the corresponding sequence for \( \delta' \). Then, abusing notation slightly, \( v^1(\delta', r') > v^1(\delta', r) > v^1(\delta, r) \), as required.

12. We now show that this type of a multi-price equilibrium exists if and only if \( \delta_1 \leq \delta < 1 \). Recall that when \( \delta = \delta_1 \), constraint 1 in (14) is binding and \( v^1 = \alpha_1 \). Because \( v^1 \) is strictly increasing in \( \delta \), if \( \delta < \delta_1 \) then \( v_1 < \alpha_1 \), leading to violation of constraint 1. Similarly, if \( \delta > \delta_1 \) then \( v^1 > \alpha_1 \) and constraint 1 is satisfied.

13. Finally, let us show that \( \delta_1 > \frac{\alpha_1 n-1}{\alpha_1(n-1)} \), as required in step 2. Suppose instead that \( \frac{\alpha_1 n-1}{\alpha_1(n-1)} > \delta_1 \). Note that the left hand side of (16) is increasing in \( \delta \), and the right hand side is independent of \( \delta \). Substituting into (16) and dividing through by \( \delta^{k-1} \), we therefore have

\[
\frac{\alpha_1 n-1}{\alpha_1(n-1)} + \frac{(1-\alpha_n)\alpha_n}{\alpha_1(n-1)} > 1.
\]

Rearranging gives \( \alpha_n + \alpha_1 > 1 + \alpha_n^2 \). Because \( \alpha_1 \) and \( \alpha_n \) are the market shares of competing firms, they must sum to no more than 1, leading to a contradiction. \( \square \)

**Proof of Lemma 4**

*Proof.* First, note that \( \alpha_n \leq 0.5 \). If \( \alpha_n \) is increased, the coefficient in front of the second term in equation (16) will increase. Given that the left hand side in increasing in \( \delta_1 \), while the right hand side is constant in \( \delta_1 \), the increase in \( \alpha_n \) will result in smaller \( \delta_1 \).

Second, if \( \alpha_1 \) is increased, the coefficient in front of the second term in equation (16) will decrease. In addition the expression on the right will increase; that is, both changes work in the same direction. Given that the left hand side in increasing in \( \delta_1 \), while the right hand side is constant in \( \delta_1 \), the increase in \( \alpha_1 \) will result in larger \( \delta_1 \). \( \square \)

**Proof of Lemma 5**

*Proof.* The proof is based on three observations. First, at the knot discount factor \( \delta_s \), \( r^s = 1 \), and constraint \( s \) is binding so that \( v^s = \alpha_1 \). Second, all constraints \( i \geq s \) are binding, so equation (33) applies. Third, for \( i \leq s \), \( r^i = 1 \). Therefore, each cartel member earns profits \( \pi^i = \frac{1-a_i}{n-1} \) for \( i \leq s \). Recursively applying (10), we have

\[
v^1 = \frac{1-a_1}{n-1} \left( 1 + \delta + \cdots + \delta^{s-2} \right) + \delta^{s-1} \alpha_1.
\]

In period \( k \), \( r^k = \alpha_n \) and \( \pi^k = \frac{\alpha_n(1-\alpha_n)}{n-1} \). Applying (10) one more time gives

\[
v^k = \frac{1-a_1}{n-1} \sum_{i=1}^{s-1} \delta^i + \delta^s \alpha_1 + \frac{\alpha_n(1-\alpha_n)}{n-1}.
\]

(34)
Combining equations (33) and (34), leads to (17). □

**Proof of Lemma 6**

*Proof.* First let \( s = k - 1 \), that is consider the range \( \delta \in [\delta_{k-1}, 1) \) and recall that there is a sales path in this range. Profits in each period \( j < k \) are therefore \( \pi^j = (1 - \alpha_1)/(n - 1) \), and profits in period \( k \) are \( \pi^k = \alpha_n(1 - \alpha_n)/(n - 1) \). Using (10), note that \( v_1 = \frac{(1-\alpha_1)(1+\delta+\cdots+\delta^{k-2})}{(1-\delta)(n-1)} \), which results in \( v_1 = \frac{(1-\alpha_1)(1-\delta^{k-1})}{(1-\delta)(n-1)} + \frac{(1-\alpha_n)\alpha_n\delta^{k-1}}{(1-\delta)(n-1)} \).

Now using a similar approach let us prove the statement for any \( s = 1, \ldots, k - 2 \). That is, consider the range \( \delta \in [\delta_s, \delta_{s+1}) \). Profits are given by \( \pi^j = \frac{1-\alpha_1}{n-1} \) for \( j \leq s \), \( \pi^j = \frac{r/(1-\alpha_1)}{n-1} \) for \( s < j < k \), and \( \pi^k = \frac{(1-\alpha_n)\alpha_n}{n-1} \). Hence,

\[
v_1 = \frac{(1-\alpha_1)(1+\delta^s+\cdots+\delta^{s+1}+\delta^{k-2}r^{s+1}+\cdots+\delta^{k-2}r^{k-1})}{(1-\delta)(n-1)} + \frac{(1-\alpha_n)\alpha_n\delta^{k-1}}{(1-\delta)(n-1)} \tag{35}
\]

Note that for \( \delta \in [\delta_s, \delta_{s+1}) \), the following condition holds: \( \alpha_n = r^k < r^{k-1} < \cdots < r^{s+1} < 1 \). From Lemma 3, it then follows that \( r^{-1} = \frac{(n-1)\delta^s+\delta(1-\alpha_n)\alpha_n}{a_1n-1} \), \( r^{-2} = \frac{(\delta a_1(n-1))/(n-1)\delta^s+\delta(1-\alpha_n)\alpha_n}{a_1n-1} \), \( r^{-s+1} = \left(\frac{\delta a_1(n-1)}{a_1n-1}\right)^{k-s-2} \frac{(n-1)\delta^s+\delta(1-\alpha_n)\alpha_n}{a_1n-1} \) \cdots. Combining these conditions with (35) results in (18), as required. □

**Proof of Lemma 7**

*Proof.* 1. First, note that \( \nu^s \) is monotonically declining in \( s \). Therefore, the relenting constraints in (15) are satisfied if and only if \( \nu^k \geq \alpha_{n-1} \). Using (10), we have \( \nu^k = \frac{\alpha_n(1-\alpha_n)}{n-1} + \delta \nu^1 \). Combining yields

\[
\nu^1 \geq \frac{(n-1)\alpha_n(1-\alpha_n)}{(n-1)\delta}.
\]

2. If \( \delta = \delta_1 \), all the undercutting constraints in (14) are binding and \( \nu^1 = \alpha_1 \). Satisfying the undercutting constraints therefore requires \( \nu^1 \geq \alpha_1 \).

3. Undercutting and relenting temptations are equated when

\[
\alpha_1 = \frac{(n-1)\alpha_{n-1} - \alpha_n(1-\alpha_n)}{(n-1)\delta}.
\]

This occurs at the discount factor

\[
\delta' = \frac{(n-1)\alpha_{n-1} - \alpha_n(1-\alpha_n)}{(n-1)\alpha_1}.
\]

4. Both undercutting and relenting constraints are binding at the same discount factor when \( \delta_1 = \delta' \). Substituting \( \delta' \) into (16) gives

\[
\frac{\alpha_{n-1}}{\alpha_1} = \left(\frac{\alpha_1 n-1}{(n-1)\alpha_{n-1} - \alpha_n(1-\alpha_n)}\right)^{k-1},
\]

29
or \( g(\alpha, k, n) = 0 \), where \( g(\alpha, k, n) \) is defined in (20). Then, \( \delta_1 > \delta' \) if and only if \( g(\alpha, k, n) < 0 \).

5. Suppose \( g(\alpha, k, n) \leq 0 \). Then, \( \delta_1 \geq \delta' \) and the undercutting constraints bind at a higher discount factor. Therefore, \( \{r^s\}_{s=1}^k \) solve (12) - (15) if and only if \( \delta \geq \delta_1 \), establishing (i).

6. Suppose \( g(\alpha, k, n) > 0 \). Then, \( \delta_1 < \delta' \) and the relenting constraints bind at a higher discount factor. Equation (19) determines the critical discount factor for the relenting constraints, \( \hat{\delta} \). If \( \delta < \hat{\delta} \), these constraints are not satisfied and the program (12) - (15) cannot be solved. If \( \delta \geq \hat{\delta} \), the constraints in (15) are satisfied. The revenue sequence solves the program (12) - (13) by assumption. Therefore, it also solves the program (12) - (15). This establishes (ii).

Proof of Lemma 8

Proof. 1. We first consider the critical discount factor \( \delta = \delta^* \). If \( \hat{\delta} \geq \delta_{k-1} \), then the cartel sets \( r^s = 1 \) for \( s = 1, \ldots, k-1 \). Therefore, the fringe undercutting in every period \( s \leq k-1 \). From now on, we consider \( \hat{\delta} < \delta_{k-1} \). Constraint \( k-1 \) is therefore binding and \( v^{k-1} = \alpha_1 r^{k-1} \) if the fringe undercutting in period \( k-1 \). Using (10), we obtain \( r^{k-1} = \frac{\delta(n-1)}{\alpha_1 n-1} v^k \).

2. Equation (22) describes a threshold revenue level, below which the cartel prefers the fringe to relent. Substituting the above relationship between \( r^{k-1} \) and \( v^k \) into (22), we obtain the left hand side of (23).

3. If \( \delta^* = \hat{\delta} \), then the relenting constraint (15) is binding for the cartel and \( v^k = \alpha_{n-1} \). Alternatively, if \( \delta^* = \delta_1 \), then undercutting constraint 1 is binding and \( v^1 = \alpha_1 \). Using (10) gives \( v^k = \frac{\delta(n-1)}{\alpha_1 n-1} + \delta_1 \alpha_1 \). Together, these results determine the value of \( v^k \) on the right hand side of (23).

4. Points 1-3 suggest that (23) provides necessary and sufficient conditions for the cartel to induce fringe undercutting in period \( k-1 \) at \( \hat{\delta} = \delta^* \). Observe that the condition on the left hand side of (23) is decreasing in \( \delta \) and \( v^k \) is increasing in \( \delta \) when \( \delta > \delta^* \). Therefore, (23) provides sufficient conditions for fringe undercutting when \( \delta > \delta^* \).

Proof of Proposition 1

Proof. (sketch)

Consider the candidate solution \( \{r^s, r^f\}_{s=1}^k \).

1. If the fringe relents, the cartel obtains a market share of \( (1-\alpha_n)/(n-1) \). If the fringe undercut, the cartel receives \( (1-\alpha_1)/(n-1) \). To induce the fringe to relent in period \( s \), the cartel must set a revenue of \( \alpha_n/\alpha_1 \). The cartel therefore prefers the fringe to undercut if

\[
\frac{1-\alpha_n}{n-1} < \frac{1-\alpha_1}{n-1} \Rightarrow r^s > \frac{\alpha_n(1-\alpha_n)}{\alpha_1(1-\alpha_1)}.
\]

2. In Lemma 3, we established the monotonicity of \( r^s \) in \( s \) in the case where the fringe undercut in all but the last period. By monotonicity, if (36) holds for \( s = k-1 \) then it must hold for \( s < k-1 \). This establishes the condition required for optimality of \( \{r^{s(1)}, r^{s(1)}_{f}\}_{s=1}^k \).
3. Next, we describe the construction of \( r^{s,(i)} \) in (24). We show the construction explicitly for the case \( i = 2 \). Construction for \( i > 2 \) follows the same steps and is omitted.

Consider \( i = 2 \). The cartel induces the fringe to undercut for \( s < k - 2 \) and induces relent for \( s > k - 2 \). Therefore \( r^{k,(2)} = \alpha_n \) and \( r^{k-1,(2)} = \alpha_n / \alpha_1 \).

Next consider period \( k - 2 \). By equation (10), \( v^k = \delta v^1 \pi^k - v^1 + \frac{(1 - \alpha_n)\alpha_n}{n - 1} \). The latter equality follows because the fringe relents in period \( k \) leaving market share \( (1 - \alpha_n)/(n - 1) \) for each cartel member, and because \( r^k = \alpha_n \). By the same logic, \( v^{k-1} = \delta v^k + \frac{(1 - \alpha_n)\alpha_n}{a_1(n - 1)} \).

Combining, we have

\[
v^{k-1} = \delta^2 v^1 + \frac{\delta(1 - \alpha_n)\alpha_n}{n - 1} + \frac{(1 - \alpha_n)\alpha_n}{\alpha_1(n - 1)}.
\] (37)

Next, consider the cartel undercutting constraint in period \( k - 2 \): \( v^{k-2} \geq \alpha_1 r^{k-2,(2)} \). Using equation (10) and noting that each cartel member obtains market share \( (1 - \alpha_1)/(n - 1) \) if the fringe undercut gives

\[
\delta v^{k-1} + \frac{1 - \alpha_1}{n - 1} r^{k-2,(2)} \geq \alpha_1 r^{k-2,(2)}.
\] (38)

Combining equations (37) and (38) and rearranging, we have

\[
r^{k-2,(2)} \leq \frac{\delta^3 v^1(n - 1) + \delta^2(1 - \alpha_n)\alpha_n + \delta(1 - \alpha_n)\alpha_n}{\alpha_1 n - 1}.
\]

Optimality of the revenue path requires \( r^{k-2,(2)} \) to be as high as possible without exceeding monopoly revenue. Therefore,

\[
r^{k-2,(2)} = \min \left\{ \frac{\delta^3 v^1(n - 1) + \delta^2(1 - \alpha_n)\alpha_n + \delta(1 - \alpha_n)\alpha_n}{\alpha_1 n - 1} \right\}.
\]

For periods \( s = 2, \ldots, k - 3 \), apply the same steps as point 5. in Lemma 3. This yields

\[
r^{s,(2)} = \min \left\{ 1, \frac{\delta \alpha_1(n - 1)}{\alpha_1 n - 1} r^{s+1,(i)} \right\}, s = 2, \ldots, k - 3.
\]

Next, show that \( r^{1,(2)} = 1 \).

4. Show/assume monotonicity of \( r^{s,(i)} \).

5. By the monotonicity of \( r^{s,(i)} \), if \( r^{s,(i)} < \alpha_n(1 - \alpha_n) / \alpha_1(1 - \alpha_1) < r^{s-1,(i)} \), then the cartel prefers to induce the fringe to undercut for all periods \( j < s \) and to relent for all periods \( j \geq s \). [We need to also show that these conditions are equivalent for all \( i \).]

6. By 2. and 5., the candidate optimal revenue stream is determined by (26).

7. For the candidate solution, we need to verify whether cartel members have an incentive to deviate. To deter undercutting, we need to show \( v^1 \geq \alpha_1 \) is required. To deter relenting, we need to show \( v^s \geq \alpha_n \). Show monotonicity of \( v^s \). Then this implies \( v^k \geq \alpha_n \).

8. Show monotonicity of \( r^{s,(i)} \) (if we assumed it earlier).
In this section, we present additional detail for Example 7. The following remark characterises the optimal 3-period cycle under the modification to Definition 1.

**Remark 1.** The 3-period cycle $\sigma^3$ solves the program (12) - (15) if and only if $\delta \geq \delta_1$. The cycle is unique. If $\delta \in [\delta_1, 1)$, the cycle has the following properties:

$$
r_1(\delta) = \begin{cases} 
\frac{\alpha_n}{\alpha_1}, & \text{if } \delta_1 \leq \delta < \delta_2, \\
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)}, & \text{if } \delta_2 \leq \delta < \delta_3, \\
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1) + a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)} & \text{if } \delta_3 \leq \delta < \delta_4, \\
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1) + a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)} & \text{if } \delta_4 \leq \delta < \delta_5, \\
1, & \text{if } \delta_5 \leq \delta < 1;
\end{cases}
$$

$$
r_2(\delta) = \begin{cases} 
\frac{\alpha_n}{\alpha_1}, & \text{if } \delta_1 \leq \delta < \delta_3, \\
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)}, & \text{if } \delta_3 \leq \delta < \delta_4, \\
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1) + a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)} & \text{if } \delta_4 \leq \delta < \delta_5, \\
1, & \text{if } \delta_5 \leq \delta < 1;
\end{cases}
$$

$$
r_3(\delta) = \alpha_n, \text{ if } \delta_1 \leq \delta < 1;
$$

$$
u(\delta) = \begin{cases} 
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)}, & \text{if } \delta_1 \leq \delta < \delta_2, \\
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1) + a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)} & \text{if } \delta_2 \leq \delta < \delta_3, \\
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1) + a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)} & \text{if } \delta_3 \leq \delta < \delta_4, \\
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1) + a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)} & \text{if } \delta_4 \leq \delta < \delta_5, \\
\frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)}{a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1) + a_1(1-\alpha_n)(\delta+a_1\delta^2 + \delta + 1)} & \text{if } \delta_5 \leq \delta < 1;
\end{cases}
$$

where the critical values $\delta_i$ satisfy $0 = \delta_1 \leq \delta_2 \leq \delta_3 \leq \delta_4 \leq \delta_5 \leq 1$ and are the unique solutions to the following cubic equations:

$$
\delta_1 : \frac{\alpha_1(1-\alpha_n)(n-1)\delta^3 + \delta^2 + \delta + 1}{1-\alpha_n} = 0,
\delta_2 : \frac{\alpha_1(1-\alpha_n)(n-1)\delta^3 + \delta^2 + \delta + 1}{1-\alpha_n} = 0,
\delta_3 : (\alpha_1(1-\alpha_n)(n-1)\delta^3 + \delta^2 + \delta + 1) - (\alpha_1(n-1))^2 = 0,
\delta_4 : (\alpha_1(1-\alpha_n)(n-1)\delta^3 + \delta^2 + \delta + 1) - (\alpha_1(n-1))^2 = 0,
\delta_5 : (\alpha_1(1-\alpha_n)(n-1)\delta^3 + \delta^2 + \delta + 1) - (\alpha_1(n-1))^2 = 0.
$$

**Proof.** First, if $\delta \in [\delta_1, \delta_2]$ the fringe always relents, consequently $r_1 = r_2 = \alpha_n / \alpha_1$ and $r_3 = \alpha_n$. That results in $\nu = \nu_1 = \frac{\alpha_n(1-\alpha_n)(1+\delta+a_1\delta^2)}{\alpha_1(1-\delta)(1+\delta+a_1\delta^2)}$.

Second, if $\delta \in [\delta_2, \delta_3]$ the fringe undercuts at $r_1$ and relents at other prices, consequently $r_2 = \alpha_n / \alpha_1$ and $r_3 = \alpha_n$. To derive $r_1$ note that $\nu_1 = \alpha_1 r_1$. That gives $r_1 = \frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2)}{\alpha_1(1-\alpha_n)(\delta+a_1\delta^2)}$ and results in $\nu = \nu_1 = \frac{\alpha_n(1-\alpha_n)(\delta+a_1\delta^2)}{\alpha_1(1-\alpha_n)(\delta+a_1\delta^2)}$.

Third, if $\delta \in [\delta_3, \delta_4]$ the fringe undercuts at $r_1$ and $r_2$ and relents at $r_3$. We also assume that all $r_i$ are less than one. Consequently $r_3 = \alpha_n$. One can derive $r_1 = \frac{\alpha_n(1-\alpha_n)(n-1)r_2}{\alpha_1(1-\alpha_n)(n-1)}$ and use...
\[ v_2 = \alpha_1 r_2 \text{ to get } r_2 = \frac{a_n(1-\alpha_n)\delta(\alpha_1 n-1)}{(\alpha_1 n-1)^2 - (\alpha_1(n-1))^2 \delta^3} \text{ and results in } v = v_1 = \frac{a_n(1-\alpha_n)\delta^2(n-1)}{(\alpha_1 n-1)^2 - (\alpha_1(n-1))^2 \delta^3} \text{ and } r_1 = \frac{a_n(1-\alpha_n)\delta^2 \alpha_1(n-1)}{(\alpha_1 n-1)^2 - (\alpha_1(n-1))^2 \delta^3}.

Fourth, if \( \delta \in [\delta_4, \delta_5] \) the fringe undercuts at \( r_1 \) and \( r_2 \) and relents at \( r_3 \); in addition, \( r_1 = 1 \) and \( r_3 = \alpha_n \). To derive \( r_2 \) note that \( v_2 = \alpha_1 r_2 \). That gives

\[
r_2 = \frac{a_n(1-\alpha_n)\delta + (1-\alpha_1)\delta^2}{\alpha_1 n-1 - \alpha_1(n-1)\delta^3}
\]

and results in \( v = v_1 = \frac{(1-\alpha_1) + (1-\alpha_1)\delta + (1-\alpha_n)\alpha_n\delta^2}{(1-\delta^3)(n-1)} \).

Fifth, if \( \delta \in [\delta_5, 1] \) the fringe undercuts at \( r_1 \) and \( r_2 \) and relents at \( r_3 \); in addition, \( r_1 = r_2 = 1 \) and \( r_3 = \alpha_n \). That results in \( v = v_1 = \frac{(1-\alpha_1) + (1-\alpha_1)\delta + (1-\alpha_n)\alpha_n\delta^2}{(1-\delta^3)(n-1)} \).

\[ \square \]

References


