Centre for Financial Risk

A New Approach for Modelling Credit Contagion

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Abstract

In this paper, we introduce a new modelling approach for credit contagion. Due to domestic and global business and financial links between lines of business/companies, a primary event, i.e. a shock which initially affects a couple of lines of business, a particular company or a region/country spreads to the rest of the lines of business, the rest of related companies and other countries. As a result, a shock to a business line, a company or a region/country can create a series of local and global default. To accommodate waves of defaults in reality from a shock such as oil and commodity price fluctuation, the governments’ fiscal and monetary policies, the release of corporate financial reports, the political and social decisions, the rumours of mergers and acquisitions among firms, collapse and bankruptcy of firms etc., we introduce a new intensity process, i.e. multiple shot noise intensity. It consists of \(d\) component processes, where each process acts a jump intensity for the next one. We use a Cox process to model the multivariate default time and derive multivariate survival/default probabilities. As an example of pricing of credit derivatives, we calculate credit default swaps (CDS) rates, assuming that jump size distributions are exponentials. Standard martingale theory is used to derive the multivariate Laplace transforms.

**Keywords:** Credit contagion; multiple shot noise process; multivariate Cox process; joint survival/default probability; conditional default probabilities; credit default swaps (CDS) rate.

1. Introduction

In financial industry, a shock which initially affects a couple of institutions or a particular region of the economy spreads to the rest of the financial industry and then infects the larger economy. This is called ‘financial contagion’ (Allen and Gale, 2000, Bae et al, 2003). The US federal takeover of *Fannie Mae* and *Freddie Mac*, the Bank of America take over of *Countrywide Financial Corporation* and the bankruptcy of *New Century Financial Corporation* due to mismanagement of subprime mortgage in US are the examples of financial contagion. The prevalence of above financial contagion has led to further bankruptcies and default of mortgage lenders in US announcing their significant losses. This subprime mortgage meltdown also has led to new ownership for *Bears Stern* and *Merrill Lynch* and the bankruptcy of *Lehman Brothers*. These contagious events have caused the collapse of stock prices in worldwide and it has shaken global financial markets further due to new waves of default and bankruptcy.

Main causes of a series of default are due to domestic and global business and financial links or ties between firms. All firms’ financial stability is universally subject to macroeconomic factors such as the price of energy and minerals, interest rates, mortgage rate and
exchange rate. Also the global economic system that allows free trades for goods & services and investment makes them highly dependent each other. As a result, a shock to a business sector or a region/country can create a series of default locally and globally. In this paper, hence we introduce a new approach for modelling financial contagion focusing on credit market, i.e. credit contagion.

During the last 5 years, the studies on credit contagion and its modelling have grown. The work by Schönbucher (2003) Giesecke and Weber (2004 and 2006), Egloff et al. (2007), Leung and Kwok (2007), Dassios and Sculli (2008), Frey and Backhaus (2008) and Herbertsson and Rootzén (2008) contain numerous models to this effect. It is now in main stream of attention as we are witnessing infectious global credit market triggered by sub-prime mortgage meltdown. One of the reasons for this contagious reaction in global credit and financial market is that collateralized debt obligations (CDOs) backed by asset-backed, e.g. CDS (credit default swap) and mortgage-backed securities have had increasing exposure to subprime mortgage bonds, i.e. junk mortgages.

To accommodate clustering of defaults in reality, we introduce a new default intensity process, i.e. multiple shot noise intensity (Dassios, 1987). It consists of $d$ component processes $\lambda^{(d)}_t$, $\lambda^{(d-1)}_t$, $\lambda^{(d-2)}_t$, $\ldots$, $\lambda^{(1)}_t$. For $i = d-1, d-2, \ldots, 1$, $\lambda^{(i)}_t$ decays with rate $\delta^{(i)} \lambda^{(i)}_t$ and additive jumps occur with rate of $\lambda^{(i+1)}_t$, i.e. each process acts a jump intensity for the next one. Jumps sizes are independent but not identically distributed random variables with distribution function $G(y^{(i)})$.

Hence the multivariate default intensity model we consider has the following structure:

$$
\begin{align*}
\alpha^{(d)}_t &= -\delta^{(d)} \lambda^{(d)}_t \, dt + dC^{(d)}_t, \\
C^{(d)}_t &= \sum_{j=1}^{M^{(d)}_t} Y^{(d)}_j,
\end{align*}
$$

$$
\begin{align*}
\lambda^{(d-1)}_t &= -\delta^{(d-1)} \lambda^{(d-1)}_t \, dt + dC^{(d-1)}_t, \\
C^{(d-1)}_t &= \sum_{k=1}^{M^{(d-1)}_t} Y^{(d-1)}_k,
\end{align*}
$$

$$
\vdots
$$

$$
\begin{align*}
\alpha^{(1)}_t &= -\delta^{(1)} \lambda^{(1)}_t \, dt + dC^{(1)}_t, \\
C^{(1)}_t &= \sum_{l=1}^{M^{(1)}_t} Y^{(1)}_l,
\end{align*}
$$

where:

- $\{Y^{(i)}_j\}_{j=1,2,\ldots}$, $\{Y^{(i)}_k\}_{k=1,2,\ldots}$, $\ldots$, $\{Y^{(i)}_l\}_{l=1,2,\ldots}$ are sequences of independent but not identically distributed random variables with distribution function $G(y^{(i)})$ ($y^{(i)} > 0$) and $i = d, d-1, \ldots, 1$.
- $M^{(i)}_t$ is the total number of events up to time $t$.
- $\delta^{(i)}$ is the rate of exponential decay for the firm $i = d, d-1, d-2, \ldots, 1$.

We also make the additional assumption that the point process $M^{(i)}_t$ and the sequences $\{Y^{(i)}\}$ are independent of each other.
$M^{(d)}_t$ follows a homogeneous Poisson process with frequency $\rho$ and $M^{(i)}_t$ for $i = d - 1$, $d - 2$, $\cdots$, 1 follows a Cox process with frequency $\lambda^{(i+1)}_t$ respectively (Cox 1955; Grandell, 1976 and Brémaud 1981). So in this model, dependence between the intensities $\lambda^{(i)}_t$ comes from the structure that each process acts a jump intensity for the next one.

The intensity $\lambda^{(d)}_t$ is triggered by primary events (or shocks) such as oil and commodity price movement, the governments’ fiscal and monetary policies, the release of corporate financial reports, the political and social decisions, the romours of mergers and acquisitions among firms, collapse and bankruptcy of firms, September 11 WTC catastrophe and Hurrican Katrina etc. that will result in a positive jump in intensity process. As time passes, the intensity process decreases, as the … rm relative local/global … rms’default intensities. As time passes, the intensity processes for event occurs which again will result in a positive jump in intensity process.

The intensity $\lambda^{(d)}_t$ is the jump arrival rate for the $(d - 1)^{th}$ firm’s default intensity $\lambda^{(d-1)}_t$ and the intensity $\lambda^{(d-1)}_t$ is the jump arrival rate for the $(d - 2)^{th}$ firm’s default intensity $\lambda^{(d-2)}_t$ and so on. Hence the intensity $\lambda^{(d)}_t$ is the prime trigger in influencing all other relative local/global firms’ default intensities. As time passes, the intensity processes for the firm $d - 1$, $d - 2$, $\cdots$ decrease, as these firms will also do their best to avoid being in bankruptcy from the influence by the prime company’s default intensity $\lambda^{(d)}_t$ that is triggered by primary events (or shocks).

We use another Cox process $N^{(i)}_t$ for $i = d, d - 1, \cdots, 1$ to model the multivariate default time and derive multivariate survival/default probabilities where it is assumed that default jumps, intensity jumps and primary event jumps are independent of each other. To obtain these, in Section 2, we start with deriving the joint Laplace transform of the vector $(\Lambda^{(d)}, \Lambda^{(d-1)}, \cdots, \Lambda^{(1)}, \lambda^{(d)}, \lambda^{(d-1)}, \cdots, \lambda^{(1)}, N^{(d)}, N^{(d-1)}, \cdots, N^{(1)}, t)$ using standard martingale theory, with which we obtain the expression for

$$
\mathbb{E}\left(e^{-\nu d \Lambda^{(d)}_t} e^{-\nu d-1 \Lambda^{(d-1)}_t} \cdots e^{-\nu 1 \Lambda^{(1)}_t} \mid \lambda^{(d)}_0, \lambda^{(d-1)}_0, \cdots, \lambda^{(1)}_0\right)
$$

where $\nu_i \geq 0$ and $\Lambda^{(i)}_t = \int_0^t \lambda^{(i)}_s ds$ for $i = d, d - 1, d - 2, \cdots, 1$. For simplicity, it is assumed that $d = 3$ but it can be easily extended to the higher dimensions. Using (1.2), in Section 3 we derive the joint survival probability, i.e.

$$
\Pr \left( \tau_3 > t, \tau_2 > t, \tau_1 > t, \mid \lambda^{(3)}_0, \lambda^{(2)}_0, \lambda^{(1)}_0 \right) = \mathbb{E}\left\{ e^{-\Lambda^{(3)}_t} e^{-\Lambda^{(2)}_t} e^{-\Lambda^{(1)}_t} \mid \lambda^{(3)}_0, \lambda^{(2)}_0, \lambda^{(1)}_0 \right\},
$$

where $\tau_i \equiv \inf \left\{ t : N^{(i)}_t = 1 \mid N^{(i)}_0 = 0 \right\}$ is the default arrival time for the firm $i$ that is equivalent to the first jump time of the Cox process $N^{(i)}_t$. The expression for the joint default probability, i.e.

$$
\Pr \left( \tau_3 \leq t, \tau_2 \leq t, \tau_1 \leq t, \mid \lambda^{(3)}_0, \lambda^{(2)}_0, \lambda^{(1)}_0 \right)
\mathbb{E}\left\{ \left(1 - e^{-\Lambda^{(3)}_t}\right) \left(1 - e^{-\Lambda^{(2)}_t}\right) \left(1 - e^{-\Lambda^{(1)}_t}\right) \mid \lambda^{(3)}_0, \lambda^{(2)}_0, \lambda^{(1)}_0 \right\}
$$

and relevant joint probabilities like
For function expression is derived, we can easily calculate them by setting to calculate the joint survival/default probability and relevant joint probabilities. Once its

\[ e^{-\Lambda_1(t)} e^{-\Lambda_2(t)} \left( 1 - e^{-\Lambda_1(t)} \right) | \lambda_0^{(2)}, \lambda_0^{(1)} \],

are omitted as they can easily be obtained using (1.2) and (1.3). We also derive conditional default probabilities in this section. In Section 4, as an example of pricing of credit derivatives, we calculate credit default swaps (CDS) rates assuming that jump size distributions are exponentials. Section 5 contains some concluding remarks.

### 2. Joint Laplace transform of the vector \((\Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, N^{(d)}, \ldots, N^{(1)}, t)\)

and joint survival probability

We firstly consider using the joint Laplace transform of the vector

\[ (\Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, N^{(d)}, \ldots, N^{(1)}, t) \]

to calculate the joint survival/default probability and relevant joint probabilities. Once its expression is derived, we can easily calculate them by setting \(\nu = 1\) in the equation (1.2).

The generator of the process \((\Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, N^{(d)}, \ldots, N^{(1)}, t)\) acting on a function \(f \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(1)}, t \right)\) belonging to its domain is given by

\[
\begin{align*}
A f & \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(1)}, t \right) \\
& = \frac{\partial f}{\partial t} + \sum_{i=1}^{d} \lambda^{(i)} \frac{\partial f}{\partial \Lambda^{(i)}} - \sum_{i=1}^{d} \delta^{(i)} \lambda^{(i)} \frac{\partial f}{\partial \lambda^{(i)}} \\
& + \sum_{i=1}^{d} \lambda^{(i)} \left[ f \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(i)}+1, \ldots, n^{(1)}, t \right) - f \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(1)}, t \right) \right] \\
& + \sum_{i=1}^{d-1} \lambda^{(i+1)} \left[ \int_{0}^{\infty} f \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)} + y^{(i)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(1)}, t \right) dG \left( y^{(i)} \right) \right. \\
& \quad \left. - f \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(1)}, t \right) \right] \\
& + \rho \left[ \int_{0}^{\infty} f \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)} + y^{(d)}, \lambda^{(d-1)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(1)}, t \right) dG \left( y^{(d)} \right) \right. \\
& \quad \left. - f \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \lambda^{(d-1)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(1)}, t \right) \right].
\end{align*}
\]

(2.1)

For \(f \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(1)}, t \right)\) to belong to the domain of the generator \(A\), it is sufficient that \(f \left( \Lambda^{(d)}, \ldots, \Lambda^{(1)}, \lambda^{(d)}, \ldots, \lambda^{(1)}, n^{(d)}, \ldots, n^{(1)}, t \right)\) is differentiable w.r.t. \(\Lambda^{(i)}, \lambda^{(i)}, t\) for all \(\Lambda^{(i)}, \lambda^{(i)}, n^{(i)}, t\) and that

\[
\left| \int_{0}^{\infty} f \left( \cdot, \lambda^{(d)} + y^{(d)}, \cdot \right) dG \left( y^{(d)} \right) - f \left( \cdot, \lambda^{(d)}, \cdot \right) \right| < \infty
\]
and

\[ \left| \int_0^\infty f \left( \cdot, \lambda^{(i)} + y^{(i)}, \cdot \right) dG \left( y^{(i)} \right) - f \left( \cdot, \lambda^{(i)}, \cdot \right) \right| < \infty \text{ for } i = d, d - 1, \ldots, 1. \]

We assume that default jumps, intensity jumps and primary event jumps do not occur at the same time.

Let us find a suitable martingale to derive the joint Laplace transform of the vector \( \left( \Lambda_t^d, \Lambda_t^{d-1}, \ldots, \Lambda_t^1 \right) \) and the joint p.g.f. (probability generating function) of the vector \( \left( N_t^d, N_t^{d-1}, \ldots, N_t^1 \right) \).

**Theorem 2.1** Considering constants \( \nu_i, k_i \) and \( \theta_i \) such that \( \nu_i \geq 0, k_i \geq 0 \) and \( 0 \leq \theta_i \leq 1 \),

\[ \prod_{i=1}^{d} e^{-\nu_i \Lambda^{(i)} t} \prod_{i=1}^{d} e^{-A_i(t) \lambda^{(i)} t} \prod_{i=1}^{d} \theta_i^{\theta^{(i)} t} e^{C(t)} \]  \hspace{1cm} (2.2)

is a martingale, where

\[ A_1(t) = \frac{\nu_1 + (1 - \theta_1)}{\delta^{(1)}} + \left\{ k_1 - \frac{\nu_1 + (1 - \theta_1)}{\delta^{(1)}} \right\} e^{\delta^{(1)} t}, \]

\[ A_i(t) = k_i e^{\delta^{(i)} t} - \nu_i \left( \frac{e^{\delta^{(i)} t} - 1}{\delta^{(i)}} \right) - (1 - \theta_i) \left( \frac{e^{\delta^{(i)} t} - 1}{\delta^{(i)}} \right) \]

\[ -e^{\delta^{(i)} t} \int_0^t e^{-\delta^{(i)} s} \left[ 1 - \hat{g}_{i-1} \{ A_{i-1}(s) \} \right] ds \text{ for } i = d, d - 1, \ldots, 2 \]

and

\[ C(t) = \rho \int_0^t \left[ 1 - \hat{g}_d \{ A_d(s) \} \right] ds, \]

where

\[ \hat{g}_i(u) = \int_0^\infty e^{-u y^{(i)}} dG(y^{(i)}). \]

**Proof.** From (2.1), \( f \left( \Lambda_t^d, \ldots, \Lambda_t^1, \lambda_t^d, \ldots, \lambda_t^1, n_t^d, \ldots, n_t^1, t \right) \) has to satisfy \( A f = 0 \) for it to be a martingale. Setting

\[ f \left( \Lambda_t^d, \Lambda_t^{d-1}, \ldots, \Lambda_t^1, \lambda_t^d, \lambda_t^{d-1}, \ldots, \lambda_t^1, n_t^d, n_t^{d-1}, \ldots, n_t^1, t \right) = \prod_{i=1}^{d} e^{-\nu_i \Lambda^{(i)} t} \prod_{i=1}^{d} e^{-A_i(t) \lambda^{(i)} t} \prod_{i=1}^{d} \theta_i^{\theta^{(i)} t} e^{C(t)} \]

we get the equation

5
\[-\sum_{i=1}^{d} \lambda^{(i)} A_{i}'(t) + C'(t) - \sum_{i=1}^{d} \lambda^{(i)} \nu_i + \sum_{i=1}^{d} \delta^{(i)} \lambda^{(i)} A_i(t) + \sum_{i=1}^{d} \lambda^{(i)} (\theta_i - 1) \]
\[+ \sum_{i=1}^{d-1} \lambda^{(i+1)} \left[ \hat{g}_i \{ A_i(t) \} - 1 \right] + \rho \left[ \hat{g}_d \{ A_d(t) \} - 1 \right] = 0. \quad (2.3)\]

Solve (2.3) and the result follows. ■

For simplicity, let us set \( d = 3 \) (i.e. \( i = 3, 2 \) and \( 1 \)).

**Theorem 2.2** Let \( \Lambda_1^{(3)}, \Lambda_1^{(2)}, \Lambda_1^{(1)}, \lambda_2^{(3)}, \lambda_2^{(2)}, \lambda_2^{(1)}, N_1^{(3)}, N_1^{(2)}, N_1^{(1)} \) as defined. Then

\[
E \left\{ \left[ e^{-\nu_1 \{ \Lambda_1^{(2)} - \lambda_2^{(1)} \} e^{-\nu_2 \{ \Lambda_1^{(3)} - \lambda_2^{(1)} \} e^{-\nu_3 \{ \Lambda_1^{(2)} - \lambda_2^{(1)} \} e^{-z_1 \lambda_2^{(1)} - \lambda_2^{(3)} \lambda_2^{(1)}}} \times \theta_1 \{ N_1^{(2)} - \lambda_2^{(1)} \} \theta_2 \{ N_1^{(3)} - \lambda_2^{(1)} \} \theta_3 \{ N_1^{(2)} - \lambda_2^{(1)} \} \right] \lambda_2^{(1)} \lambda_2^{(2)} \lambda_2^{(3)} \right\}
\]
\[= \exp \left[ - \left( \frac{\nu_1 + (1 - \theta_1)}{\delta^{(1)}} \right) + \left( \frac{\nu_1 + (1 - \theta_1)}{\delta^{(1)}} \right) e^{-\delta^{(1)}(t_2 - t_1)} \right] \lambda_2^{(1)} \lambda_2^{(2)} \lambda_2^{(3)} \]
\[\times \exp \left[ -B(\zeta_2, \zeta_1, \nu_2, \nu_1, \theta_2, \theta_1, t_2, t_1) \lambda_2^{(3)} \right] \]
\[\times \exp \left[ -D(\zeta_3, \zeta_2, \zeta_1, \nu_3, \nu_2, \nu_1, \theta_3, \theta_2, \theta_1, t_2, t_1) \lambda_2^{(3)} \right], \quad (2.4)\]

where

\[
B(\zeta_2, \zeta_1, \nu_2, \nu_1, \theta_2, \theta_1, t_2, t_1) = \zeta_2 e^{-\delta^{(2)}(t_2 - t_1)} + \nu_2 \left( \frac{1 - e^{-\delta^{(2)}(t_2 - t_1)}}{\delta^{(2)}} \right) + \left( 1 - \theta_2 \right) \left( \frac{1 - e^{-\delta^{(2)}(t_2 - t_1)}}{\delta^{(2)}} \right) + e^{-\delta^{(2)}(t_2 - t_1)} \int_0^{t_2 - t_1} \exp \left[ \left( 1 - \frac{\nu_1 + (1 - \theta_1)}{\delta^{(1)}} \right) + \left( \frac{\nu_1 + (1 - \theta_1)}{\delta^{(1)}} \right) e^{-\delta^{(1)}s} \right] ds,
\]

\[
C(\zeta_3, \zeta_2, \zeta_1, \nu_3, \nu_2, \nu_1, \theta_3, \theta_2, \theta_1, t_2, t_1) = \left[ \begin{array}{c}
\zeta_3 e^{-\delta^{(2)}(t_2 - t_1)} + \nu_3 \left( \frac{1 - e^{-\delta^{(2)}(t_2 - t_1)}}{\delta^{(3)}} \right) \\
+ (1 - \theta_3) \left( \frac{1 - e^{-\delta^{(2)}(t_2 - t_1)}}{\delta^{(3)}} \right) \\
+ e^{\delta^{(3)}t_1} \int_0^{t_2 - t_1} e^{-\delta^{(3)}(t_2 - s)} ds
\end{array} \right] \times \left[ \begin{array}{c}
\zeta_2 e^{-\delta^{(2)}s} + \nu_2 \left( \frac{1 - e^{-\delta^{(2)}s}}{\delta^{(2)}} \right) + \left( 1 - \theta_2 \right) \left( \frac{1 - e^{-\delta^{(2)}s}}{\delta^{(2)}} \right) + e^{-\delta^{(2)}s} \int_0^{s} \exp \left[ \left( 1 - \frac{\nu_1 + (1 - \theta_1)}{\delta^{(1)}} \right) + \left( \frac{\nu_1 + (1 - \theta_1)}{\delta^{(1)}} \right) e^{-\delta^{(1)}u} \right] du \end{array} \right] ds,
\]

\[
D(\zeta_3, \zeta_2, \zeta_1, \nu_3, \nu_2, \nu_1, \theta_3, \theta_2, \theta_1, t_2, t_1) = -\rho
\]
Proof. The results follow immediately if we set 

$$
\zeta_3 e^{-\delta(3)s} + \nu_3 \left( \frac{1-e^{-\delta(3)s}}{\delta(3)} \right) + (1-\theta_3) \zeta_3 \left( \frac{1-e^{-\delta(3)s}}{\delta(3)} \right) + e^{-\delta(3)s}
$$

and

$$
0 < t_1 < t_2, \quad \zeta_i \geq 0 \quad \text{for} \quad i = 3, 2, 1.
$$

Corollary 2.3 Let $\nu, \zeta$ and $\theta$ as defined for $i = 3, 2$ and $1$. Then
are given by

\[
E \left\{ e^{-\nu_1 \{ \Lambda_t^{(1)} - \Lambda_t^{(1)} \}} e^{-\nu_2 \{ \Lambda_t^{(2)} - \Lambda_t^{(2)} \}} e^{-\nu_3 \{ \Lambda_t^{(3)} - \Lambda_t^{(3)} \}} \times e^{\zeta_1 \lambda_t^{(1)}} e^{\zeta_2 \lambda_t^{(2)}} e^{\zeta_3 \lambda_t^{(3)}} \mid \lambda_t^{(1)}, \lambda_t^{(2)}, \lambda_t^{(3)} \right\}
\]

\[
= \exp \left[ - \left( \frac{\nu_1}{\delta^{(1)}} + \left( \zeta_1 - \frac{\nu_1}{\delta^{(1)}} \right) e^{-\delta^{(1)}(t_2-t_1)} \right) \lambda_t^{(1)} \right]
\times \exp \left[ -B(\zeta_2, \zeta_1, 0, 0, 0, \nu_1, 1, 1, t_2, t_1) \lambda_t^{(2)} \right]
\times \exp \left[ -C(\zeta_3, \zeta_2, 0, 0, 0, \nu_1, 1, 1, 1, t_2, t_1) \lambda_t^{(3)} \right]
\times \exp \left[ D(\zeta_3, \zeta_2, 0, 0, 0, \nu_1, 1, 1, 1, t_2, t_1) \right]
\]

\[(2.5)\]

and

\[
E \left\{ \theta_1^{N_t^{(1)} - N_t^{(1)}} \theta_2^{N_t^{(2)} - N_t^{(2)}} \theta_3^{N_t^{(3)} - N_t^{(3)}} \times e^{-\zeta_1 \lambda_t^{(1)}} e^{-\zeta_2 \lambda_t^{(2)}} e^{-\zeta_3 \lambda_t^{(3)}} \mid \lambda_t^{(1)}, \lambda_t^{(2)}, \lambda_t^{(3)} \right\}
\]

\[
= \exp \left[ - \left( \frac{1 - \theta_1}{\delta^{(1)}} + \left( \zeta_1 - \frac{1 - \theta_1}{\delta^{(1)}} \right) e^{-\delta^{(1)}(t_2-t_1)} \right) \lambda_t^{(1)} \right]
\times \exp \left[ -B(\zeta_2, 0, 0, 0, 0, 0, \theta_2, 0, t_2, t_1) \lambda_t^{(2)} \right]
\times \exp \left[ -C(\zeta_3, 0, 0, 0, 0, 0, \theta_3, \theta_2, t_2, t_1) \lambda_t^{(3)} \right]
\times \exp \left[ D(\zeta_3, 0, 0, 0, 0, \theta_3, \theta_2, t_2, t_1) \right].
\]

\[(2.6)\]

**Proof.** If we set \( \theta_3 = \theta_2 = \theta_1 = 1 \) in (2.4), (2.5) follows. If we also set \( \nu_3 = \nu_2 = \nu_1 = 0 \) in (2.4), (2.6) follows. \( \square \)

Now we can easily derive the joint Laplace transform of the vector \( \left( \Lambda_t^{(3)}, \Lambda_t^{(2)}, \Lambda_t^{(1)} \right) \), the joint Laplace transform of the vector \( \left( \lambda_t^{(3)}, \lambda_t^{(2)}, \lambda_t^{(1)} \right) \) and the joint p.g.f. of the vector \( \left( \theta_t^{(3)}, \theta_t^{(2)}, \theta_t^{(1)} \right) \).

**Corollary 2.4** The joint Laplace transform of the vector \( \left( \Lambda_t^{(3)}, \Lambda_t^{(2)}, \Lambda_t^{(1)} \right) \) and \( \left( \lambda_t^{(3)}, \lambda_t^{(2)}, \lambda_t^{(1)} \right) \) are given by

\[
E \left\{ e^{-\nu_1 \{ \Lambda_t^{(1)} - \Lambda_t^{(1)} \}} e^{-\nu_2 \{ \Lambda_t^{(2)} - \Lambda_t^{(2)} \}} e^{-\nu_3 \{ \Lambda_t^{(3)} - \Lambda_t^{(3)} \}} \times \lambda_t^{(1)}, \lambda_t^{(2)}, \lambda_t^{(3)} \right\}
\]

\[
= \exp \left[ - \left( \frac{\nu_1}{\delta^{(1)}} - \frac{\nu_1}{\delta^{(1)}} \right) \lambda_t^{(1)} \right]
\times \exp \left[ -B(0, 0, 0, 0, 0, 0, 0, 0, 0, t_2, t_1) \lambda_t^{(2)} \right]
\times \exp \left[ -C(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, t_2, t_1) \lambda_t^{(3)} \right]
\times \exp \left[ D(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, t_2, t_1) \right].
\]

\[(2.7)\]
\[ \begin{align*}
&= \exp \left[ - \left( \frac{1 - \theta_1}{\delta(1)} \right) + \left( \zeta_1 - \frac{1 - \theta_1}{\delta(1)} \right) e^{-\delta(1)(t_2 - t_1)} \lambda_{t_1}^{(1)} \right] \\
&\times \exp \left[ -B(0, 0, 0, 0, \theta_2, \theta_1, t_2, t_1) \lambda_{t_1}^{(2)} \right] \\
&\times \exp \left[ -C(0, 0, 0, 0, 0, \theta_3, \theta_2, \theta_1, t_2, t_1) \lambda_{t_1}^{(3)} \right] \\
&\times \exp \left[ D(0, 0, 0, 0, 0, \theta_3, \theta_2, \theta_1, t_2, t_1) \right].
\end{align*} \]

and the joint p.g.f. of the vector \((\theta_{t_1}^{(3)}, \theta_{t_1}^{(2)}, \theta_{t_1}^{(1)})\) is given by

\[ \begin{align*}
&= \exp \left[ - \left( \frac{1 - \theta_1}{\delta(1)} \right) + \left( \zeta_1 - \frac{1 - \theta_1}{\delta(1)} \right) e^{-\delta(1)(t_2 - t_1)} \lambda_{t_1}^{(1)} \right] \\
&\times \exp \left[ -B(0, 0, 0, 0, \theta_2, \theta_1, t_2, t_1) \lambda_{t_1}^{(2)} \right] \\
&\times \exp \left[ -C(0, 0, 0, 0, 0, \theta_3, \theta_2, \theta_1, t_2, t_1) \lambda_{t_1}^{(3)} \right] \\
&\times \exp \left[ D(0, 0, 0, 0, 0, \theta_3, \theta_2, \theta_1, t_2, t_1) \right].
\end{align*} \]

**Proof.** If we set \(\zeta_3 = \zeta_2 = \zeta_1 = 0\) in (2.5) and (2.6) respectively, (2.7) and (2.9) follow. If we also set \(\nu_3 = \nu_2 = \nu_1 = 0\) in (2.5) or set \(\theta_3 = \theta_2 = \theta_1 = 1\) in (2.6), (2.8) follows. □

**Corollary 2.5** The joint Laplace transform of the vector \((\lambda_{t}^{(3)}, \lambda_{t}^{(2)}, \lambda_{t}^{(1)})\) where \(\lambda_{t}^{(3)}, \lambda_{t}^{(2)}\) and \(\lambda_{t}^{(1)}\) are jointly stationary is given by

\[ \begin{align*}
&= \exp \left[ - \left( \frac{1 - \theta_1}{\delta(1)} \right) + \left( \zeta_1 - \frac{1 - \theta_1}{\delta(1)} \right) e^{-\delta(1)(t_2 - t_1)} \lambda_{t_1}^{(1)} \right] \\
&\times \exp \left[ -B(0, 0, 0, 0, \theta_2, \theta_1, t_2, t_1) \lambda_{t_1}^{(2)} \right] \\
&\times \exp \left[ -C(0, 0, 0, 0, 0, \theta_3, \theta_2, \theta_1, t_2, t_1) \lambda_{t_1}^{(3)} \right] \\
&\times \exp \left[ D(0, 0, 0, 0, 0, \theta_3, \theta_2, \theta_1, t_2, t_1) \right].
\end{align*} \]

**Proof.** Let \(t_2 \to \infty\) in (2.8) and the result follows. □

3. Joint/marginal survival probability and conditional default probabilities
Having derived the joint Laplace transform of the vector \( (\Lambda_t^{(3)}, \Lambda_t^{(2)}, \Lambda_t^{(1)}) \), the joint Laplace transform of the vector \( (\Lambda_t^{(3)}, \Lambda_t^{(2)}, \lambda_t^{(1)}) \) and the joint p.g.f. of the vector \( (\theta_t^{(3)}, \theta_t^{(2)}, \theta_t^{(1)}) \) in the previous section, we can easily calculate the joint/marginal survival probability and the conditional default probabilities. Using (2.10), we have the joint Laplace transform of the vector \( (\Lambda_t^{(3)}, \Lambda_t^{(2)}, \lambda_t^{(1)}) \) where \( \lambda_t^{(3)}, \lambda_t^{(2)} \) and \( \lambda_t^{(1)} \) are jointly stationary, i.e.

\[
\mathcal{E} \left\{ e^{-\nu_3 \{\Lambda_t^{(3)} - \Lambda_t^{(1)}\}} e^{-\nu_2 \{\Lambda_t^{(2)} - \Lambda_t^{(1)}\}} e^{-\nu_1 \{\Lambda_t^{(1)} - \Lambda_t^{(1)}\}} \right\} = \exp[-J(\nu_3, \nu_2, \nu_1, t_2, t_1)] \times \exp[D(0, 0, 0, \nu_3, \nu_2, \nu_1, 1, 1, 1, t_2, t_1)],
\]

(3.1)

where

\[
J(\nu_3, \nu_2, \nu_1, t_2, t_1) = -\rho
\]

\[
\times \int_0^\infty \left[ 1 - \frac{\nu_3}{\delta^{(3)}} \right] \left[ 1 - \frac{\nu_2}{\delta^{(2)}} \right] \left[ 1 - \frac{\nu_1}{\delta^{(1)}} \right] ds
\]

(3.2)

If we set \( \nu_3 = \nu_2 = \nu_1 = 1 \) in (3.1), we can easily obtain the joint survival probability for which the stationary distribution of the default intensity is used, i.e.

\[
\mathcal{E} \left\{ e^{-\{\Lambda_t^{(3)} - \Lambda_t^{(1)}\}} e^{-\{\Lambda_t^{(2)} - \Lambda_t^{(1)}\}} e^{-\{\Lambda_t^{(1)} - \Lambda_t^{(1)}\}} \right\} = \exp[-J(1, 1, t_2, t_1)] \times \exp[D(0, 0, 1, 1, 1, 1, 1, t_2, t_1)].
\]

(3.2)

We omit the expressions for the joint default probability and other relevant joint probabilities as they can easily be obtained using (1.4), (1.5) and (3.2).

To get the explicit expressions, now let us consider for \( i = 3, 2 \) and use an exponential distribution of \( G(y^{(3)}) \) and \( G(y^{(2)}) \) for jump size respectively, i.e.
\[ g(y^{(3)}) = \alpha e^{-\alpha y^{(3)}} \quad \text{and} \quad g(y^{(2)}) = \beta e^{-\beta y^{(2)}} \quad \text{with} \quad \alpha > 0, \quad \beta > 0, \]

then setting \( \nu_3 = \nu_2 = 1 \) and \( \nu_1 = 0 \) in (3.1), the corresponding joint survival probability of firm 3 and 2 is given by
\[
E \left\{ e^{-\Lambda^{(3)}_2 - \Lambda^{(3)}_1} e^{-\Lambda^{(2)}_2 - \Lambda^{(2)}_1} \right\}
\]
\[
= \exp \left[ -\rho \int_0^{t_2-t_1} \left[ \frac{1-e^{-\delta(t_2-t_1)}}{\delta(t_2-t_1)} + e^{-\delta(t_2-t_1)} \int_0^s e^{\delta(t_2-t_1)} z \left[ \frac{1-e^{-\delta(t_2-t_1)}}{\beta \delta(t_2-t_1)} \right] dz \right] ds \right]
\]
\[
\times \exp \left[ -\rho \int_0^{\infty} \left[ \frac{1-e^{-\delta(t_2-t_1)}}{\delta(t_2-t_1)} \right. \right.
\left. + e^{-\delta(t_2-t_1)} \int_0^{t_2-t_1} e^{-\delta(t_2-t_1)} \left[ \frac{1-e^{-\delta(t_2-t_1)}}{\beta \delta(t_2-t_1)} \right] dz \right] ds \right].
\]

Similarly, we can easily derive the corresponding joint survival probability of firm 3 and 1, i.e. \( E \left\{ e^{-\Lambda^{(3)}_2 - \Lambda^{(3)}_1} e^{-\Lambda^{(2)}_2 - \Lambda^{(2)}_1} \right\} \) and the joint survival probability of firm 2 and 1, i.e. \( E \left\{ e^{-\Lambda^{(2)}_2 - \Lambda^{(2)}_1} e^{-\Lambda^{(3)}_2 - \Lambda^{(3)}_1} \right\} \). If we set \( \nu_2 = \nu_1 = 0 \) and \( \nu_3 = 1 \) in (3.1) and use an exponential distribution for jump sizes, i.e. \( g(y^{(3)}) = \alpha e^{-\alpha y^{(3)}} \), then the survival probability of firm 3 which is the seeding company causing series of default, is given by
\[
E \left\{ e^{-\Lambda^{(3)}_2 - \Lambda^{(3)}_1} \right\} = \frac{\alpha e^{-\delta(t_2-t_1)} \left\{ \frac{\delta(t_2-t_1)}{\delta(t_2-t_1)} \right\}^{\delta(t_2-t_1)}}{\alpha + \frac{1}{\delta(t_2-t_1)} \left( 1 - e^{-\delta(t_2-t_1)} \right)} \]
\[
= \frac{\alpha e^{-\delta(t_2-t_1)}}{\alpha + \frac{1}{\delta(t_2-t_1)} \left( 1 - e^{-\delta(t_2-t_1)} \right)} \left( 1 - e^{-\delta(t_2-t_1)} \right)^{\delta(t_2-t_1) - 1},
\]

which can be found in Dassios and Jang (2003). If we set \( \nu_3 = \nu_1 = 0 \) and \( \nu_2 = 1 \) in (3.1) and use an exponential distribution for jump sizes, i.e. \( g(y^{(2)}) = \beta e^{-\beta y^{(2)}} \), the survival probability of firm 2 is given by
If we set \( \nu_3 = \nu_2 = 0 \) and \( \nu_1 = 1 \) in (3.1) and use an exponential distribution for jump sizes, i.e. \( g(y^{(1)}) = \gamma e^{-\gamma y^{(1)}} \) \((\gamma > 0)\), the survival probability of firm 1 is given by

\[
E \left\{ e^{-\Lambda^{(1)}_{t_2} - \Lambda^{(1)}_{t_1}} \right\}
= \exp
\]
Using Bayes' rule, the conditional default probabilities between firm 3 and 2, denoted by
\( p_{3|2} \) and \( p_{2|3} \) can be calculated by

\[
p_{3|2} = \frac{p_{32}}{p_2} = \frac{\Pr(\tau_3 \leq t, \tau_2 \leq t)}{\Pr(\tau_2 \leq t)} = \frac{1 - \mathbb{E}\left\{e^{-\Lambda_2^{(3)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_1^{(2)}}\right\} + \mathbb{E}\left\{e^{-\Lambda_3^{(3)}}e^{-\Lambda_1^{(2)}}\right\}}{1 - \mathbb{E}\left\{e^{-\Lambda_2^{(3)}}\right\}},
\]

\[
p_{2|3} = \frac{p_{32}}{p_3} = \frac{\Pr(\tau_3 \leq t, \tau_2 \leq t)}{\Pr(\tau_3 \leq t)} = \frac{1 - \mathbb{E}\left\{e^{-\Lambda_2^{(3)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_2^{(2)}}\right\} + \mathbb{E}\left\{e^{-\Lambda_3^{(3)}}e^{-\Lambda_2^{(2)}}\right\}}{1 - \mathbb{E}\left\{e^{-\Lambda_3^{(3)}}\right\}}
\]

and other conditional default probabilities can be easily driven using (3.2).

Now let us illustrate the calculations of the joint survival/default probabilities, relevant joint probabilities and the conditional default probabilities.

**Example 3.1**

We assume that the magnitude of the contribution to the default intensity of the firm 3 from the primary events is higher than that of the firm 2. We also assume that the decay rate for the firm 3, that measures how quick the firm gets out of the influence of primary events lowering their default intensity rate, is lower than that for the firm 2. So the parameter values used to calculate the joint probabilities are

\[
\alpha = 5, \quad \beta = 10, \quad \delta^{(3)} = 0.3, \quad \delta^{(2)} = 0.5 \quad \text{and} \quad \rho = 4.
\]

Setting \( t_1 = 0 \) and \( t_2 = 1 \), joint survival probability is given by

\[
\Pr\left(\tau_3 > 1, \tau_2 > 1\right) = \mathbb{E}\left\{e^{-\Lambda_2^{(3)}}e^{-\Lambda_1^{(2)}}\right\}
\]

and relevant joint probabilities are given by

\[
\Pr\left(\tau_3 \leq 1, \tau_2 > 1\right) = \mathbb{E}\left\{e^{-\Lambda_2^{(2)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_3^{(3)}}e^{-\Lambda_1^{(2)}}\right\}
\]

\[
\Pr\left(\tau_3 > 1, \tau_2 \leq 1\right) = \mathbb{E}\left\{e^{-\Lambda_3^{(3)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_3^{(3)}}e^{-\Lambda_1^{(2)}}\right\},
\]

and the joint default probability is given by

\[
\Pr\left(\tau_3 \leq 1, \tau_2 \leq 1\right) = 1 - \mathbb{E}\left\{e^{-\Lambda_1^{(3)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_2^{(2)}}\right\} + \mathbb{E}\left\{e^{-\Lambda_3^{(3)}}e^{-\Lambda_2^{(2)}}\right\}.
\]

From the equation (3.3)-(3.5), the calculations of the joint survival/default probability, relevant joint probabilities and the conditional default probability are given by
\[
\begin{align*}
\Pr(\tau_3 > 1, \tau_2 > 1) &= 0.05492 \\
\Pr(\tau_3 \leq 1, \tau_2 > 1) &= 0.54808, \\
\Pr(\tau_3 > 1, \tau_2 \leq 1) &= 0.03137, \\
\Pr(\tau_3 \leq 1, \tau_2 \leq 1) &= 0.36563 \tag{3.7}
\end{align*}
\]

and

\[
p_{3\mid 2} = 0.92098 \quad \text{and} \quad p_{2\mid 3} = 0.40016, \tag{3.8}
\]

where

\[
\begin{align*}
\Pr(\tau_3 > 1) &= 0.08629 \quad \text{and} \quad \Pr(\tau_2 > 1) = 0.603, \tag{3.9} \\
\Pr(\tau_3 \leq 1) &= 0.91371 \quad \text{and} \quad \Pr(\tau_2 \leq 1) = 0.397. \tag{3.10}
\end{align*}
\]

The joint survival probability is given by 0.05492 which is much lower than the survival probability of the firm 2, 0.603 and lower than the survival probability of the firm 3, 0.08629. The joint probabilities where the survivorship of the firm 2 is concerned are determined by the survival or default probability of the firm 3, i.e. as the default probability of the firm 3 is higher than its survival probability we have

\[
\Pr(\tau_3 \leq 1, \tau_2 > 1) = 0.54808 > \Pr(\tau_3 > 1, \tau_2 > 1) = 0.05492.
\]

Similarly, the joint default probability is given by 0.36563 which is much lower than the default probability of the firm 3, 0.91371 and lower than the default probability of the firm 2, 0.397. The joint probabilities where the defaultability of the firm 2 is concerned are determined by the survival or default probability of the firm 3, i.e. as the default probability of the firm 3 is higher than its survival probability we have

\[
\Pr(\tau_3 \leq 1, \tau_2 \leq 1) = 0.36563 > \Pr(\tau_3 > 1, \tau_2 \leq 1) = 0.03137.
\]

The conditional default probabilities are rescaled to the joint default probability by the default probability of the firm 3 and 2 in (3.10) respectively. Hence the default probabilities of the firm 3 (or 2) given that the firm 2 (or 3) defaults in (3.8) are very similar to the unconditional default probability of the firm 3 (or 2) in (3.10).

Next examples shows that the calculations of the joint survival/default probability, relevant joint probabilities and the conditional default probability by changing the values of the jump frequency rate \( \rho \), the values of the magnitude of jump size \( \alpha \) and the values of exponential decay \( \delta^{(3)} \) respectively.

**Example 3.2**

Using the same parameter values used in Example 3.1 except for \( \rho = 10, \alpha = 1 \) and \( \delta^{(3)} = 0.01 \), the calculations of the joint survival/default probability, relevant joint probabilities and the conditional default probability are given in Table 3.1.
Table 3.1.
\[
\begin{array}{|c|c|c|}
\hline
\rho = 10 & \alpha = 1 & \delta^{(3)} = 0.01 \\
\hline
\Pr (\tau_3 > 1, \tau_2 > 1) & \Pr (\tau_3 > 1, \tau_2 > 1) & \Pr (\tau_3 > 1, \tau_2 > 1) \\
= 0.00070684 & = 0.000016267 & = 0 \\
\hline
\Pr (\tau_3 \leq 1, \tau_2 > 1) & \Pr (\tau_3 \leq 1, \tau_2 > 1) & \Pr (\tau_3 \leq 1, \tau_2 > 1) \\
= 0.2816532 & = 0.089082733 & = 0 \\
\hline
\Pr (\tau_3 > 1, \tau_2 \leq 1) & \Pr (\tau_3 > 1, \tau_2 \leq 1) & \Pr (\tau_3 > 1, \tau_2 \leq 1) \\
= 0.0014804 & = 0.00061367 & = 0 \\
\hline
\Pr (\tau_3 \leq 1, \tau_2 \leq 1) & \Pr (\tau_3 \leq 1, \tau_2 \leq 1) & \Pr (\tau_3 \leq 1, \tau_2 \leq 1) \\
= 0.7161596 & = 0.910839633 & = 1 \\
\hline
p_{3|2} = 0.99794 & p_{3|2} = 0.99993 & p_{3|2} = 1 \\
\text{and } p_{2|3} = 0.71773 & \text{and } p_{2|3} = 0.91091 & \text{and } p_{2|3} = 1 \\
\hline
\Pr (\tau_3 > 1) = 0.0021872 & \Pr (\tau_3 > 1) = 0.000077634 & \Pr (\tau_3 > 1) = 0 \\
\text{and } \Pr (\tau_2 > 1) = 0.28236 & \text{and } \Pr (\tau_2 > 1) = 0.089099 & \text{and } \Pr (\tau_2 > 1) = 0 \\
\hline
\Pr (\tau_3 \leq 1) = 0.9978128 & \Pr (\tau_3 \leq 1) = 0.999922366 & \Pr (\tau_3 \leq 1) = 1 \\
\text{and } \Pr (\tau_2 \leq 1) = 0.71764 & \text{and } \Pr (\tau_2 \leq 1) = 0.910901 & \text{and } \Pr (\tau_2 \leq 1) = 1 \\
\hline
\end{array}
\]

Firstly, Table 3.1 shows that the higher the jump arrival rate in firm 3 default intensity $\lambda^{(3)}$, $\rho$ is, i.e. the more frequent arrival of primary events is, the joint/marginal survival probabilities are getting closer to 0. Secondly it shows that the lower $\alpha$ is, i.e. the bigger the magnitude of the contribution to the default intensity of the firm 3 from the primary events is, the joint/marginal survival probabilities are getting closer to 0. Lastly we can see that the lower $\delta^{(3)}$ is, i.e. the slower the firm gets out of the influence of primary events, the joint/marginal survival probabilities are getting closer to 0.

We omit the corresponding calculations for the joint survival/default probability, relevant joint probabilities and the conditional default probability by changing the values of the magnitude of jump size $\beta$ and the values of exponential decay $\delta^{(2)}$ as they can be easily obtained. It is obvious that the lower $\beta$ and the lower $\delta^{(3)}$ is, the joint/marginal survival probabilities are getting closer to 0.

4. Measuring credit default swaps (CDS) rate

In this Section, we apply the results in Section 3 in pricing of a financial product. For that purpose, we choose credit default swaps (CDS) as three parties are involved in this contract, i.e. a reference credit, a CDS buyer and a CDS seller. In order to calculate credit default swaps (CDS) rate, firstly we assume that deterministic instantaneous rate of interest $r$ for a zero-coupon default-free bond. Then its price at time 0, paying $1$ at time $t$ is given by

\[
B(0, t) = e^{-rt},
\]

where $B(0, t)$ denotes the price of a default-free zero-coupon bond.

Now we denote the default intensity process of a CDS buyer and seller by $\lambda_t^{(b)}$ and $\lambda_t^{(s)}$, respectively. We also specify the default intensity process of a reference credit by $\lambda_t^{(rc)}$. For simplicity, we assume a deterministic recovery rate $\pi$. Then credit default swaps (CDS)
rate, denoted by $\bar{s}$, is given by

$$
\bar{s} = (1 - \pi) \frac{\sum_{k=1}^{k_N} e^{rc,s}(0, t_{k-1}, t_k)}{\sum_{n=1}^{N} (t_{k_n+1} - t_{k_n}) B^b(0, t_{k_n})},
$$

where

$$
e^{rc,s}(0, t_{k-1}, t_k) = \mathbb{E} \left[ \exp \left( - \int_0^{t_k} r_s ds \right) \left\{ \exp \left( - \int_0^{t_{k-1}} \lambda^{(rc)}_s ds \right) - \exp \left( - \int_0^{t_k} \lambda^{(rc)}_s ds \right) \right\} \times \left\{ \exp \left( - \int_0^{t_k} \lambda^{(s)}_s ds \right) \right\} \mid \lambda^{(rc)}_0, \lambda^{(s)}_0 \right],
$$

(4.3)

$$
B^b(0, t_{k_n}) = \mathbb{E} \left[ \exp \left( - \int_0^{t_{k_n}} (r_s + \lambda^{(b)}_s) ds \right) \mid \lambda^{(b)}_0 \right],
$$

(4.4)

and $t_{k_1} < t_{k_2} < \cdots < t_{k_n}$. Assuming that $\lambda^{(i)}_t$ is stationary, the equation (4.3) and (4.4) can be expressed as

$$
e^{rc,s}(0, t_{k-1}, t_k) = e^{-rt_k} \times \left[ \mathbb{E} \left( e^{-\Lambda^{(rc)}_{t_{k-1}} - \Lambda^{(s)}_{t_{k-1}}} \right) \mathbb{E} \left\{ \exp \left( - \int_{t_{k-1}}^{t_k} \lambda^{(s)}_s ds \right) \right\} - \mathbb{E} \left( e^{-\Lambda^{(rc)}_{t_k} - \Lambda^{(s)}_{t_k}} \right) \right] = e^{-rt_k} \times \left[ \mathbb{E} \left( e^{-\Lambda^{(rc)}_{t_{k-1}} - \Lambda^{(s)}_{t_{k-1}}} \right) \mathbb{E} \left\{ e^{-\left( \Lambda^{(s)}_{t_{k-1}} - \Lambda^{(s)}_{t_k} \right)} \right\} - \mathbb{E} \left( e^{-\Lambda^{(rc)}_{t_k} - \Lambda^{(s)}_{t_k}} \right) \right],
$$

(4.5)

and

$$
B^b(0, t_{k_n}) = e^{-rt_{k_n}} \mathbb{E} \left( e^{-\Lambda^{(b)}_{t_{k_n}}} \right).
$$

(4.6)

Using the equation (4.6), we may consider pricing defaultable bonds as well as credit spread between default-free bond and defaultable bond. Also based on the results in Section 2 and 3, i.e. using multiple shot noise intensity, we can calculate the joint survival probability between the reference credit and CDS seller and the survival probability of CDS buyer in various ways.

Now to illustrate the calculations of credit default swaps (CDS) rates, let us assume that the default intensity process of reference credit follows $\lambda^{(3)}_t$ with $\rho$, $\alpha^{(rc)}$ and $\delta^{(rc)}$, i.e.

$$
\lambda^{(3)}_t = \lambda^{(rc)}_t \text{ with } \rho, \alpha^{(rc)} \text{ and } \delta^{(rc)}
$$
and that the default intensity process of CDS seller follows \( \lambda_t^{(2)} \) with \( \beta^{(s)} \) and \( \delta^{(s)} \), i.e.

\[
\lambda_t^{(2)} = \lambda_t^{(s)} \quad \text{with} \quad \beta^{(s)} \quad \text{and} \quad \delta^{(s)}
\]

and that the default intensity process of CDS buyer follows \( \lambda_t^{(2)} \) with \( \beta^{(b)} \) and \( \delta^{(b)} \), i.e.

\[
\lambda_t^{(2)} = \lambda_t^{(b)} \quad \text{with} \quad \beta^{(b)} \quad \text{and} \quad \delta^{(b)}
\]

The above assumptions mean that firstly the default intensity of a CDS seller and of a CDS buyer are contaminated by the default intensity of a reference credit which is the seeding company, respectively. Secondly, the defaultability (or survivorship) of a reference credit and the defaultability (or survivorship) of a CDS seller are jointly related and they are not independent each other. Also the defaultability (or survivorship) of a reference credit and the defaultability (or survivorship) of a CDS buyer are dependent each other.

**Example 4.1**

We assume that the magnitude of the contribution to the default intensity of a reference credit from the primary events is bigger than the magnitude of a CDS seller’s default intensity and of a CDS buyer’s intensity, respectively. The magnitude of a CDS seller’s default intensity is bigger than that of a CDS buyer’s intensity. It is also assumed that the decay rate in a reference credit’s intensity, that measures how quick a reference credit gets out of the influence of primary events lowering its default intensity rate, is lower than the decay rate in a CDS seller’s intensity and the decay rate in a CDS buyer’s intensity, respectively. The decay rate in a CDS seller’s intensity is lower than that of in a CDS buyer’s intensity. So the parameter values used to calculate (4.2) are

\[
\alpha^{(rc)} = 5, \quad \beta^{(s)} = 10, \quad \beta^{(b)} = 20, \quad \delta^{(rc)} = 0.3, \quad \delta^{(s)} = 0.5, \quad \delta^{(b)} = 1 \quad \text{and} \quad \rho = 4. \tag{4.7}
\]

and

\[
r = 0.05, \quad \pi = 50\%, \quad N = 2, \quad t_{k_0} = 0, \quad t_{k_1} = 0.5 \quad \text{and} \quad t_{k_2} = 1. \tag{4.8}
\]

For \( \mathbb{E} \left( e^{-\Lambda_t^{(rc)}} e^{-\Lambda_t^{(s)}} \right) \) in (4.5) and for \( \mathbb{E} \left( e^{-\Lambda_t^{(b)}} \right) \) in (4.6), (3.3) and (3.5) are used respectively, for which using the same parameter values above, the calculations of credit default swaps (CDS) rates by changing in the values of \( \rho, \alpha^{(rc)} \) and \( \delta^{(rc)} \) for a reference credit are shown in Table 4.1-4.3.

<table>
<thead>
<tr>
<th>Table 4.1.</th>
<th>Table 4.2.</th>
<th>Table 4.3.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho = 50 )</td>
<td>( \bar{s} = 18.364 ) bp</td>
<td>( \alpha^{(rc)} = 1 )</td>
</tr>
<tr>
<td>( \rho = 30 )</td>
<td>( \bar{s} = 175.31 ) bp</td>
<td>( \alpha^{(rc)} = 5 )</td>
</tr>
<tr>
<td>( \rho = 10 )</td>
<td>( \bar{s} = 1633.2 ) bp</td>
<td>( \alpha^{(rc)} = 10 )</td>
</tr>
<tr>
<td>( \rho = 4 )</td>
<td>( \bar{s} = 3126.2 ) bp</td>
<td>( \alpha^{(rc)} = 15 )</td>
</tr>
<tr>
<td>( \rho = 1 )</td>
<td>( \bar{s} = 2809.9 ) bp</td>
<td>( \alpha^{(rc)} = 20 )</td>
</tr>
</tbody>
</table>

In Table 4.1, firstly we can see that if the primary event arrival rate \( \rho \) is getting higher, the credit swap rate is getting higher. It is obvious as a CDS seller should charge higher
rate when the primary event arrivals are more frequent. However, the credit swap rate starts
decreasing due to contagion. It happens as the higher the primary event arrival rate, \( \rho \)
is, the more likely for a reference credit to default and the higher the jump arrival rate of \( \lambda_t^{(rc)} \)
for a credit swap seller and for a credit swap buyer, respectively. As a result, a credit swap
seller’s default intensity is getting higher and its increase is much bigger than the increase of
credit swap buyer’s default intensity due to the parameter values used in (4.7). Eventually
it may lead to a credit swap seller’s default which happens quicker than a credit swap buyer’s
default. The credit swap rates are getting lower as a credit swap seller’s default intensity is
getting higher due to higher the jump arrival rate of \( \lambda_t^{(rc)} \) for a credit swap seller.

Secondly, Table 4.2 shows that if the magnitude of contribution of primary event \( i \) to
intensity \( \lambda_i^{(rc)} \) is getting higher (i.e. the lower \( \alpha^{(rc)} \) value is), the credit swap rate is getting
higher as the more likely for the reference credit to default. However similar to Table 4.1,
the credit swap rate is getting lower due to contagion. It is as the lower \( \alpha^{(rc)} \) value is, the
more likely for a reference credit to default and the higher the jump arrival rate of \( \lambda_t^{(rc)} \)
for a credit swap seller and for a credit swap buyer, respectively. The swap rates decreases as
a credit swap seller’s default intensity is getting higher due to higher the jump arrival rate
of \( \lambda_t^{(rc)} \) for a credit swap seller.

Lastly, it is shown that the slower a reference credit gets out of the influence from primary
events (i.e. the lower \( \delta^{(rc)} \) value is), the credit swap fee is getting higher in Table 4.3. However
similar to Table 4.1-4.2, the credit swap fee decreases due to contagion. It happens as the
lower \( \delta^{(rc)} \) value is, the more likely for a reference credit to default and the higher the jump
arrival rate of \( \lambda_t^{(rc)} \) for a credit swap seller and for a credit swap buyer, respectively. The swap rates are getting lower as a credit swap seller no longer able to guarantee the payment
to a credit swap buyer due to a credit swap seller’s high default intensity.

Let us now examine the effect on credit default swaps (CDS) rate caused by changes in
the value of \( \beta^{(s)} \) and \( \delta^{(s)} \) for a CDS seller.

**Example 4.2**

Using the same parameter values of (4.7) and (4.8), the calculations of credit default
swaps (CDS) rates caused by changes in the value of \( \beta^{(s)} \) and \( \delta^{(s)} \) for a CDS seller are shown
in Table 4.4 and Table 4.5, respectively.

**Table 4.4.**

<table>
<thead>
<tr>
<th>( \beta^{(s)} )</th>
<th>( \bar{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>125.71bp</td>
</tr>
<tr>
<td>5</td>
<td>1921.2bp</td>
</tr>
<tr>
<td>10</td>
<td>3126.2bp</td>
</tr>
<tr>
<td>15</td>
<td>3719.2bp</td>
</tr>
<tr>
<td>20</td>
<td>4066.1bp</td>
</tr>
</tbody>
</table>

**Table 4.5.**

<table>
<thead>
<tr>
<th>( \delta^{(s)} )</th>
<th>( \bar{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0bp</td>
</tr>
<tr>
<td>0.1</td>
<td>428.53bp</td>
</tr>
<tr>
<td>0.3</td>
<td>2238.0bp</td>
</tr>
<tr>
<td>0.5</td>
<td>3126.2bp</td>
</tr>
<tr>
<td>1</td>
<td>4034.7bp</td>
</tr>
</tbody>
</table>

Table 4.4 and 4.5 show that CDS rate is getting lower to 0 by decreasing the value of \( \beta^{(s)} \)
and \( \delta^{(s)} \) for the CDS seller. It is as the lower \( \beta^{(s)} \) value and the lower \( \delta^{(s)} \) value is respectively,
the more likely the CDS seller defaults and the lower CDS rate becomes. Hence it is getting
cheaper for a CDS buyer to purchase a CDS contract, however a CDS buyer should bear in
mind that the lower the CDS rate is, the more likely a CDS seller defaults. The worst case
scenario for a CDS buyer is when both a reference credit and a CDS seller default.
The next example examines the effect on credit default swaps (CDS) rate caused by changes in the value of \(\beta(b)\) and \(\delta(b)\) for a CDS buyer.

**Example 4.3**

Using the same parameter values of (4.7) and (4.8), the calculations of credit default swaps (CDS) rates caused by changes in the value of \(\beta(b)\) and \(\delta(b)\) for a CDS buyer are shown in Table 4.6 and Table 4.7, respectively.

<table>
<thead>
<tr>
<th>(\beta(b))</th>
<th>(\bar{s})</th>
<th>(\delta(b))</th>
<th>(\bar{s})</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4081.4bp</td>
<td>0.1</td>
<td>7101.2bp</td>
</tr>
<tr>
<td>10</td>
<td>3430.2bp</td>
<td>0.3</td>
<td>3898.5bp</td>
</tr>
<tr>
<td>15</td>
<td>3225.9bp</td>
<td>0.5</td>
<td>3439.2bp</td>
</tr>
<tr>
<td>20</td>
<td>3126.2bp</td>
<td>1</td>
<td>3126.2bp</td>
</tr>
</tbody>
</table>

Table 4.6 and 4.7 show that CDS rate is getting higher by decreasing the value of \(\beta(b)\) and \(\delta(b)\) for the CDS buyer. It is as the lower \(\beta(b)\) value and the lower \(\delta(b)\) value is respectively, the more likely the CDS buyer defaults and the higher CDS rate becomes. Hence the fee payments made by a CDS buyer to a CDS seller is getting higher, however a CDS seller should bear in mind that the higher the CDS rate is, the more likely a CDS buyer defaults. A worst case scenario for a CDS seller is when a CDS buyer defaults soon after a CDS contract is initiated as there are no more fee payments made by a CDS buyer.

5. **Conclusion**

We used multiple shot noise default intensity, where each process acts a jump intensity for the next one, to model credit contagion. Using a Cox process to model the multivariate default time with the assumption that default jumps, intensity jumps and primary event jumps are independent of each other and standard martingale theory, the expressions for the joint/marginal survival probability and conditional default probabilities and their calculations were shown. For the application of these expressions, the calculations of credit default swaps (CDS) rates were presented as three parties are involved in this contract where we used exponential distributions for jump sizes. We also examined the relationship between CDS rates and the parameter values in default intensity for the reference credit, for the CDS seller and for CDS buyer respectively.

Neither linear correlation nor copula-based non-linear correlation used in this paper. On the contrary, we used multivariate intensity-based framework, i.e. a multivariate Cox process with multiple shot noise intensity to accommodate counterparty risks and to price CDS considering credit contagion from primary events. We leave the pricing of credit portfolio derivatives such as CDOs for further research based on our modeling approach.

**References**


