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And Its Application to CDS rate

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Copula-dependent collateral default intensity and its application to CDS rate

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Abstract
In this paper, we consider a correlation between the default intensities to incorporate dependency between multivariate Cox process. Assuming that each obligor has its own default intensity process, we use multivariate shot noise intensity process where jumps (i.e. magnitude of contribution of primary events to default intensities) occur collaterally and their sizes are correlated. A homogeneous Poisson process is used to describe collateral event jumps in default intensities. The Farlie-Gumbel-Morgenstern (FGM) copulas with exponential margins are used to produce correlations between event jump sizes. Applying copula-dependent collateral default intensity to multivariate Cox process, we derive the joint Laplace transform that provides us with joint survival/default probability and other relevant joint probabilities. As an example of using joint survival/default probability, we calculate credit default swaps (CDS) rates assuming that a zero-coupon default-free bond price is deterministic. Standard martingale theory is used to derive the joint Laplace transforms.

Keywords: Multivariate shot noise process; multivariate Cox process; joint survival/default probability; the Farlie-Gumbel-Morgenstern copulas; conditional default probabilities; credit default swaps (CDS) rate.

1. Introduction

Over the recent years, numerous papers have looked at the modelling for dependence of default intensities via a Cox process or a point process (Schönbucher and Schubert 2001; Jouanin et al. 2001; Yu 2005 and Giesecke 2006). Besides the construction of a point process, considerable attraction is given to the use of copulas to measure default dependence between the obligors (Li 2000; Schönbucher and Schubert 2001; Jouanin et al. 2001 and Giesecke 2004). The possibility to incorporate default dependence between multiple firms is to introduce a correlation between their intensity processes. The work by Duffie and Singleton (1998) considered joint jumps in the default intensity. Kijima (2000) and Jarrow & Yu (2001) developed it further considering the possibility of default-event triggers that cause joint default. Another approach is based on credit contagion which is from the previous research by Davis and Lo (2000, 2001). In practice, once one firm defaults, it causes an increase in other firms’ default intensities accordingly due to business links or ties between firms. A couple of works for default intensities based on credit contagion can be found in Schönbucher (2003) Giesecke (2004) and Giesecke and Weber (2006).

This paper is mainly based on the first approach and it is for the case when the firms in the complementary or substitute industry/sector are hit by a common external event, hence the word “collateral” in the title. Hence for the purpose of this paper, we concentrate on a very specific vector of intensity process, i.e multivariate shot noise process where these intensities are triggered by primary events such as oil and commodity prices, the governments’ fiscal and monetary policies, the release of corporate financial reports, the political and social decisions, the rumours of mergers and acquisitions among firms, collapse and bankruptcy of firms, September 11 WTC catastrophe and Hurricane Katrina etc. that will result in positive jumps collaterally in intensity processes. As time passes, default intensity processes decrease respectively, as all firms in the market will do their
best to avoid being in bankruptcy after the arrival of a primary event. This decrease continues until
another event occurs which again will result in positive jumps collaterally in intensity processes. The multivariate Cox process (Cox 1955; Grandell, 1976 and Brémaud 1981) is used to model the
joint default time.

With these model specification, in Section 2 we derive the joint Laplace transform using standard martingale theory that leads us to the joint survival/default probability and other relevant joint probabilities. For the dependence between the vector of event jumps, we use a Farlie-Gumbel-Morgenstern copula. We illustrate these joint probabilities and conditional default probabilities in Section 3. In Section 4, as an application of using joint survival/default probability, we calculate credit default swaps (CDS) rates assuming that zero-coupon default-free bond price is deterministic. We also assume that a deterministic recovery rate for simplicity. Section 5 contains some concluding remarks.

2. Joint Laplace transforms

The multivariate collateral default intensity model we consider has the following structure:

\[ d\lambda_t^{(1)} = -\delta^{(1)}\lambda_t^{(1)} \, dt + dL_t^{(1)}, \quad L_t^{(1)} = \sum_{j=1}^{M_t} X_j^{(1)}, \]
\[ d\lambda_t^{(2)} = -\delta^{(2)}\lambda_t^{(2)} \, dt + dL_t^{(2)}, \quad L_t^{(2)} = \sum_{j=1}^{M_t} X_j^{(2)}, \]
\[ \vdots \]
\[ d\lambda_t^{(n)} = -\delta^{(n)}\lambda_t^{(n)} \, dt + dL_t^{(n)}, \quad L_t^{(n)} = \sum_{j=1}^{M_t} X_j^{(n)}, \quad (2.1) \]

where:

- \( \{X_j^{(1)}, X_j^{(2)}, \ldots, X_j^{(n)}\}_{j=1,2,\ldots} \) is a vector sequence of dependent but not identically distributed random variables with distribution function \( F^{(i)}(x) \) (\( x > 0 \)) and \( i = 1, 2, \ldots, n \).
- \( M_t \) is the total number of events up to time \( t \).
- \( \delta^{(i)} \) is the rate of exponential decay for firm \( i = 1, 2, \ldots, n \).

We also make the additional assumption that the point process \( M_t \) and the vector sequence \( \{X_j^{(1)}, X_j^{(2)}, \ldots, X_j^{(n)}\}_{j=1,2,\ldots} \) are independent of each other.

In this model, the dependence between the intensities \( \lambda_t^{(i)} \) comes from the common event arrival process \( M_t \), together with the dependence between the vector of jumps \( \{X_j^{(1)}, X_j^{(2)}, \ldots, X_j^{(n)}\} \). The latter is modelled using the notion of copula (Nelson, 1998 and McNeil et al., 2005), i.e. the joint distribution of the vector \( \{X_j^{(1)}, X_j^{(2)}, \ldots, X_j^{(n)}\} \) is assumed to be of the form \( C(F^{(1)}, F^{(2)}, \ldots, F^{(n)}) \) with a given copula \( C \).

We assume that \( n = 2 \) for simplicity and as a specific example for \( C \), we use the Farlie-Gumbel-Morgenstern copulas, which are given by

\[ C(u, v) = uv + \theta uv(1-u)(1-v), \quad (2.2) \]
where $u \in [0,1]$, $v \in [0,1]$ and $\theta \in [-1,1]$. To make later calculation somewhat easier, we also assume that $F^{(1)}(x) = F(x_1) = 1 - e^{-\alpha x_1}$ ($\alpha > 0$, $x_1 > 0$) and $F^{(2)}(x) = F(x_2) = 1 - e^{-\beta x_2}$ ($\beta > 0$, $x_2 > 0$). The resulting joint distribution function $F(x_1,x_2)$ takes the form:

$$F(x_1,x_2) = 1 - e^{-\beta x_2} - e^{-\alpha x_1} + e^{-\alpha x_1 - \beta x_2} + \theta e^{-\alpha x_1 - \beta x_2} - \theta e^{-2\alpha x_1 - \beta x_2} - \theta e^{-\alpha x_1 - 2\beta x_2} + \theta e^{-2\alpha x_1 - 2\beta x_2}.$$  

(2.3)

It is assumed to be of this form for a matter of convenience, i.e. closed-form expressions of final results are easily derived. It is of interest to use other copulas with other margins in modelling dependence between the vector of jumps $(X_j^{(1)}, X_j^{(2)}, \cdots, X_j^{(n)})$. However as it is not possible to obtain the explicit expressions of final results using different copulas and margins from those used in this paper, numerical methods need to be used to calculate the final results.

We assume that event arrival process $M_t$, i.e. collateral jump process follows a homogeneous Poisson process with frequency $\rho$. Other point processes can be considered if the frequency of event arrival is not deterministic. To model the joint default time, multivariate Cox process (Cox 1955; Grandell, 1976 and Brémaud 1981) is used. This type of processes has been used by Gasper and Schmidt (2005, 2007, 2008 and 2009). Other point processes can also be considered modelling the joint default time beyond Cox processes.

To calculate the joint survival/default probability and relevant joint probabilities, we firstly consider deriving the joint Laplace transform of the vector $(\Lambda_t^{(1)}, \Lambda_t^{(2)})$, i.e.

$$E\left(e^{-\gamma \Lambda_t^{(1)}} e^{-\xi \Lambda_t^{(2)}} \mid \lambda_0^{(1)}, \lambda_0^{(2)}\right)$$

(2.4)

where $\gamma \geq 0$, $\xi \geq 0$ and $\Lambda_t^{(i)} = \int_0^t \lambda_s^{(i)} ds$ ($i = 1,2$). Once this is derived, we can easily calculate the joint survival probability setting $\gamma = 1$ and $\xi = 1$ in the equation (2.4), i.e.

$$\Pr \left( \tau_1 > t, \tau_2 > t \mid \lambda_0^{(1)}, \lambda_0^{(2)} \right) = E \left\{ e^{-\Lambda_t^{(1)}} e^{-\Lambda_t^{(2)}} \mid \lambda_0^{(1)}, \lambda_0^{(2)} \right\}$$

(2.5)

and relevant joint probabilities like

$$\Pr \left( \tau_1 > t, \tau_2 \leq t \mid \lambda_0^{(1)}, \lambda_0^{(2)} \right) = E \left\{ e^{-\Lambda_t^{(1)}} \left(1 - e^{-\Lambda_t^{(2)}}\right) \mid \lambda_0^{(1)}, \lambda_0^{(2)} \right\},$$

(2.6)

$$\Pr \left( \tau_1 \leq t, \tau_2 > t \mid \lambda_0^{(1)}, \lambda_0^{(2)} \right) = E \left\{ \left(1 - e^{-\Lambda_t^{(1)}}\right) e^{-\Lambda_t^{(2)}} \mid \lambda_0^{(1)}, \lambda_0^{(2)} \right\}$$

(2.7)

and the joint default probability, i.e.

$$\Pr \left( \tau_1 \leq t, \tau_2 \leq t \mid \lambda_0^{(1)}, \lambda_0^{(2)} \right) = E \left\{ \left(1 - e^{-\Lambda_t^{(1)}}\right) \left(1 - e^{-\Lambda_t^{(2)}}\right) \mid \lambda_0^{(1)}, \lambda_0^{(2)} \right\},$$

(2.8)

where $\tau^{(i)} \equiv \inf \left\{ t : N_t^{(i)} = 1 \mid N_0^{(i)} = 0 \right\}$ is the default arrival time for the firm $i = 1,2$ that is equivalent to the first jump time of the Cox process $N_t^{(i)}$ ($i = 1,2$) respectively.

The generator of the process $(\Lambda_t^{(1)}, \Lambda_t^{(2)}, \lambda_t^{(1)}, \lambda_t^{(2)}, t)$ acting on a function $f \left( \Lambda^{(1)}, \Lambda^{(2)}, \lambda^{(1)}, \lambda^{(2)}, t \right)$ belonging to its domain is given by
\[ A f \left( \Lambda^{(1)}, \Lambda^{(2)}, \lambda^{(1)}, \lambda^{(2)}, t \right) = \frac{\partial f}{\partial t} + \lambda^{(1)} \frac{\partial f}{\partial \Lambda^{(1)}} - \delta^{(1)} \lambda^{(1)} \frac{\partial f}{\partial \Lambda^{(1)}} + \lambda^{(2)} \frac{\partial f}{\partial \Lambda^{(2)}} - \delta^{(2)} \lambda^{(2)} \frac{\partial f}{\partial \Lambda^{(2)}} \]

\[ + \rho \left[ \int_{0}^{\infty} \int_{0}^{\infty} f \left( \Lambda^{(1)}, \Lambda^{(2)}, \lambda^{(1)} + x_{1}, \lambda^{(2)} + x_{2}, t \right) \frac{\partial^{2}C(F_{\chi^{(1)}(x_{1})}, F_{\chi^{(2)}(x_{2})})}{\partial x_{1} \partial x_{2}} dx dy \right] \frac{\partial f}{\partial \Lambda^{(1)}} \frac{\partial f}{\partial \Lambda^{(2)}} \]

where \( \frac{\partial^{2}C(F(x), F(y))}{\partial x \partial y} \) is the joint density of even jump sizes. For \( f \left( \Lambda^{(1)}, \Lambda^{(2)}, \lambda^{(1)}, \lambda^{(2)}, t \right) \) to belong to the domain of the generator \( A \), it is sufficient that \( f \left( \Lambda^{(1)}, \Lambda^{(2)}, \lambda^{(1)}, \lambda^{(2)}, t \right) \) is differentiable w.r.t. \( \Lambda^{(1)}, \Lambda^{(2)}, \lambda^{(1)}, \lambda^{(2)}, t \) for all \( \Lambda^{(1)}, \Lambda^{(2)}, \lambda^{(1)}, \lambda^{(2)}, t \) and that

\[ \left| \int_{0}^{\infty} \int_{0}^{\infty} f \left( \cdot, \lambda^{(1)} + x_{1}, \lambda^{(2)} + x_{2}, \cdot \right) \frac{\partial^{2}C(F_{\chi^{(1)}(x_{1})}, F_{\chi^{(2)}(x_{2})})}{\partial x_{1} \partial x_{2}} dx dy - f \left( \cdot, \lambda^{(1)}, \lambda^{(2)}, \cdot \right) \right| < \infty. \]

Now let us find a suitable martingale to derive the joint Laplace transform of the vector \( \left( \Lambda^{(1)}_{t}, \Lambda^{(2)}_{t}, \lambda^{(1)}_{t}, \lambda^{(2)}_{t} \right) \) at time \( t \).

**Theorem 2.1** Considering constants \( \gamma \geq 0, \xi \geq 0, \nu \geq 0 \) and \( \eta \geq 0 \), then

\[ \exp \left( -\gamma \Lambda^{(1)}_{t} \right) \exp \left( -\xi \Lambda^{(2)}_{t} \right) \times \exp \left[ - \left\{ \frac{\gamma}{\delta^{(1)}} + \nu - \frac{\gamma}{\delta^{(1)}} \right\} \lambda^{(1)}_{t} \right] \times \exp \left[ - \left\{ \frac{\xi}{\delta^{(2)}} + \eta - \frac{\xi}{\delta^{(2)}} \right\} \lambda^{(2)}_{t} \right] \times \exp \left[ \rho \int_{0}^{t} \left[ 1 - \hat{c} \left\{ \frac{\gamma}{\delta^{(1)}} + \nu - \frac{\gamma}{\delta^{(1)}} \right\} \lambda^{(1)}_{s} + \left\{ \frac{\xi}{\delta^{(2)}} + \eta - \frac{\xi}{\delta^{(2)}} \right\} \lambda^{(2)}_{s} \right] ds \right] \]

is a martingale where \( \hat{c}(\zeta, \varphi) = \int_{0}^{\infty} \int_{0}^{\infty} e^{-\xi x_{1} - \varphi x_{2}} \rho \partial^{2}C(F_{\chi^{(1)}(x_{1})}, F_{\chi^{(2)}(x_{2})}) dx dy \).

**Proof.** From (*) \( f \left( \Lambda^{(1)}, \Lambda^{(2)}, \lambda^{(1)}, \lambda^{(2)}, t \right) \) has to satisfy \( Af = 0 \) for it to be a martingale. Setting \( f = e^{-\gamma \Lambda^{(1)}_{t}} e^{-\xi \Lambda^{(2)}_{t}} e^{-A_{1}(t)} e^{-A_{2}(t)} e^{R(t)} \) we get the equation

\[ -\lambda^{(1)} A'_{1} (t) - \lambda^{(2)} A'_{2} (t) + R' (t) - \lambda^{(1)} \gamma - \lambda^{(2)} \xi + \delta^{(1)} \lambda^{(1)} A_{1} (t) + \delta^{(2)} \lambda^{(2)} A_{2} (t) + \rho \left[ \hat{c} \left\{ A_{1} (t), A_{2} (t) \right\} - 1 \right] = 0 \]

and solving it we get

\[ A_{1} (t) = \frac{\gamma}{\delta^{(1)}} + \left( \nu - \frac{\gamma}{\delta^{(1)}} \right) e^{\delta^{(1)} t}, \quad A_{2} (t) = \frac{\xi}{\delta^{(2)}} + \left( \eta - \frac{\xi}{\delta^{(2)}} \right) e^{\delta^{(2)} t} \quad \text{and} \]

\[ R (t) = \rho \int_{0}^{t} \left[ 1 - \hat{c} \left\{ \frac{\gamma}{\delta^{(1)}} + \nu - \frac{\gamma}{\delta^{(1)}} \right\} e^{\delta^{(1)} s} + \left\{ \frac{\xi}{\delta^{(2)}} + \eta - \frac{\xi}{\delta^{(2)}} \right\} e^{\delta^{(2)} s} \right] ds. \]
Hence the result follows.

Using a martingale in Theorem 2.1, we can easily obtain the joint Laplace transform of the vector \( \left( \Lambda_t^{(1)}, \Lambda_t^{(2)}, \Lambda_t^{(1)}, \Lambda_t^{(2)} \right) \) at time \( t \).

**Corollary 2.2** Considering constants \( \kappa \geq 0 \) and \( \psi \geq 0 \), the joint Laplace transform of the vector \( \left( \Lambda_t^{(1)}, \Lambda_t^{(2)}, \Lambda_t^{(1)}, \Lambda_t^{(2)} \right) \) is given by

\[
\mathbb{E} \left\{ e^{-\gamma (\Lambda_{t_2}^{(1)} - \Lambda_{t_1}^{(1)})} e^{-\xi (\Lambda_{t_2}^{(2)} - \Lambda_{t_1}^{(2)})} e^{-\kappa \lambda_{t_2}^{(1)}} e^{-\psi \lambda_{t_2}^{(2)}} \mid \lambda_{t_1}^{(1)}, \lambda_{t_1}^{(2)} \right\}
\]

\[
= \exp \left\{ \frac{\gamma}{\delta^{(1)}} \left[ 1 - e^{-\delta^{(1)}(t_2-t_1)} \right] \lambda_{t_1}^{(1)} \right\} \times \exp \left\{ \frac{\xi}{\delta^{(2)}} \left[ 1 - e^{-\delta^{(2)}(t_2-t_1)} \right] \lambda_{t_1}^{(2)} \right\} \times \exp \left\{ -\rho \int_{t_1}^{t_2} \left[ 1 - \frac{\gamma}{\delta^{(1)}} \left( 1 - e^{-\delta^{(1)} s} \right), \frac{\xi}{\delta^{(2)}} \left( 1 - e^{-\delta^{(2)} s} \right) \right] ds \right\},
\]

where \( t_2 > t_1 \).

**Proof.** Set \( \kappa = \frac{\gamma}{\delta^{(1)}} + \nu - \gamma (t_2-t_1) e^{\delta^{(1)} t_2} \) and \( \psi = \frac{\xi}{\delta^{(2)}} + \left( \eta - \frac{\xi}{\delta^{(2)}} \right) e^{\delta^{(2)} t_2} \) in Theorem 1, the result follows immediately.

Let us find out the expression of the joint Laplace transform of the vector \( \left( \Lambda_t^{(1)}, \Lambda_t^{(2)} \right) \) and its corresponding expression using the joint distribution function \( F(x_1, x_2) \) driven by (2.2).

**Corollary 2.3** The joint Laplace transform of the vector \( \left( \Lambda_t^{(1)}, \Lambda_t^{(2)} \right) \) is given by

\[
\mathbb{E} \left\{ e^{-\gamma (\Lambda_{t_2}^{(1)} - \Lambda_{t_1}^{(1)})} e^{-\xi (\Lambda_{t_2}^{(2)} - \Lambda_{t_1}^{(2)})} \mid \lambda_{t_1}^{(1)}, \lambda_{t_1}^{(2)} \right\}
\]

\[
= \exp \left\{ \frac{\gamma}{\delta^{(1)}} \left[ 1 - e^{-\delta^{(1)}(t_2-t_1)} \right] \lambda_{t_1}^{(1)} \right\} \times \exp \left\{ \frac{\xi}{\delta^{(2)}} \left[ 1 - e^{-\delta^{(2)}(t_2-t_1)} \right] \lambda_{t_1}^{(2)} \right\} \times \exp \left\{ -\rho \int_{t_1}^{t_2} \left[ 1 - \frac{\gamma}{\delta^{(1)}} \left( 1 - e^{-\delta^{(1)} s} \right), \frac{\xi}{\delta^{(2)}} \left( 1 - e^{-\delta^{(2)} s} \right) \right] ds \right\},
\]

and setting (2.3) in (2.12), its corresponding expression is given by

\[
\mathbb{E} \left\{ e^{-\gamma (\Lambda_{t_2}^{(1)} - \Lambda_{t_1}^{(1)})} e^{-\xi (\Lambda_{t_2}^{(2)} - \Lambda_{t_1}^{(2)})} \mid \lambda_{t_1}^{(1)}, \lambda_{t_1}^{(2)} \right\}
\]

\[
= \exp \left\{ \frac{\gamma}{\delta^{(1)}} \left[ 1 - e^{-\delta^{(1)}(t_2-t_1)} \right] \lambda_{t_1}^{(1)} \right\} \times \exp \left\{ \frac{\xi}{\delta^{(2)}} \left[ 1 - e^{-\delta^{(2)}(t_2-t_1)} \right] \lambda_{t_1}^{(2)} \right\} \times \exp \left\{ -\rho \int_{t_1}^{t_2} \left[ \frac{A(s, \xi) + B(s, \gamma) + C(s, \gamma, \xi) \cdot D(s, \gamma) F(s, \xi) - G(s, \gamma, \xi) \cdot H(s, \gamma) I(s, \xi) D(s, \gamma) F(s, \xi)}{H(s, \gamma) I(s, \xi) D(s, \gamma) F(s, \xi)} \right] ds \right\},
\]

where

\[
A(s, \xi) = \frac{\alpha \xi (1 - e^{-\delta^{(2)} s})}{\delta^{(2)}}, \quad B(s, \gamma) = \frac{\beta \gamma (1 - e^{-\delta^{(1)} s})}{\delta^{(1)}},
\]
\[ C(s, \gamma, \xi) = \frac{\gamma \xi \left(1 - e^{-\delta(1)s}\right) \left(1 - e^{-\delta(2)s}\right)}{\delta(1) \delta(2)}, \]

\[ D(s, \gamma) = 2\alpha + \frac{\gamma \left(1 - e^{-\delta(1)s}\right)}{\delta(1)}, \quad F(s, \xi) = 2\beta + \frac{\xi \left(1 - e^{-\delta(2)s}\right)}{\delta(2)}, \]

\[ G(s, \gamma, \xi) = \frac{\theta \alpha \beta \gamma \xi \left(1 - e^{-\delta(1)s}\right) \left(1 - e^{-\delta(2)s}\right)}{\delta(1) \delta(2)}, \]

\[ H(s, \gamma) = \alpha + \frac{\gamma \left(1 - e^{-\delta(1)s}\right)}{\delta(1)}, \quad I(s, \xi) = \beta + \frac{\xi \left(1 - e^{-\delta(2)s}\right)}{\delta(2)}. \]

**Proof.** (2.12) follows immediately if we set \( \kappa = 0 \) and \( \psi = 0 \) in (2.11) from which (2.13) can be found using (2.3) as the joint distribution of the vector \((X_j^{(1)}, X_j^{(2)})\). ■

**Corollary 2.4** Using the asymptotic (stationary) distribution of \( \lambda_t \) in Dassios and Jang (2003), (2.13) is given by

\[\begin{align*}
E \left\{ e^{-\gamma (\Lambda_{t_2}^{(1)} - \Lambda_{t_1}^{(1)})} e^{-\xi (\Lambda_{t_2}^{(2)} - \Lambda_{t_1}^{(2)})} \right\} &= \left\{ \frac{\alpha}{\alpha + \frac{\gamma}{\delta(1)} \left(1 - e^{-\delta(1)(t_2-t_1)}\right)} \right\} \left\{ \frac{\beta}{\beta + \frac{\xi}{\delta(2)} \left(1 - e^{-\delta(2)(t_2-t_1)}\right)} \right\} \times \exp \left[ -\rho \int_{t_1}^{t_2} \frac{\{A(s, \xi) + B(s, \gamma) + C(s, \gamma, \xi)\} D(s, \gamma) F(s, \xi) - G(s, \gamma, \xi)}{H(s, \gamma) I(s, \xi) D(s, \gamma) F(s, \xi)} \, ds \right].
\end{align*}\]  

**(2.14)**

**Proof.** From Theorem 2.6 of Dassios and Jang (2003), the result follows. ■

3. **Joint survival/default probabilities and their calculations**

Using the results in Section 2, we can easily derive the joint survival/default probability and other relevant joint probabilities. Throughout the rest of this paper, we use (2.14) as the joint Laplace transform of the vector \((\Lambda_t^{(1)}, \Lambda_t^{(2)})\).

**Lemma 3.1** The joint survival probability is given by

\[\begin{align*}
E \left\{ e^{-\gamma (\Lambda_{t_2}^{(1)} - \Lambda_{t_1}^{(1)})} e^{-\xi (\Lambda_{t_2}^{(2)} - \Lambda_{t_1}^{(2)})} \right\} &= \left\{ \frac{\alpha}{\alpha + \frac{1}{\delta(1)} \left(1 - e^{-\delta(1)(t_2-t_1)}\right)} \right\} \left\{ \frac{\beta}{\beta + \frac{1}{\delta(2)} \left(1 - e^{-\delta(2)(t_2-t_1)}\right)} \right\} \times \exp \left[ -\rho \int_{t_1}^{t_2} \frac{\{A(s) + B(s) + C(s)\} D(s) F(s) - G(s)}{H(s) I(s) D(s) F(s)} \, ds \right],
\end{align*}\]  

where

\[ \gamma, \xi, \alpha, \beta, \delta(1), \delta(2) \in (0, 1) \]
\[ A(s) = \frac{\alpha \left(1 - e^{-\delta(2)s}\right)}{\delta(2)}, \quad B(s) = \frac{\beta \left(1 - e^{-\delta(1)s}\right)}{\delta(1)}, \]
\[ C(s) = \frac{\left(1 - e^{-\delta(1)s}\right) \left(1 - e^{-\delta(2)s}\right)}{\delta(1)\delta(2)}, \]
\[ D(s) = 2\alpha + \frac{\left(1 - e^{-\delta(1)s}\right)}{\delta(1)}, \quad F(s) = 2\beta + \frac{\left(1 - e^{-\delta(2)s}\right)}{\delta(2)}, \]
\[ G(s) = \frac{\theta \alpha \beta \left(1 - e^{-\delta(1)s}\right)\left(1 - e^{-\delta(2)s}\right)}{\delta(1)\delta(2)}, \]
\[ H(s) = \alpha + \frac{\left(1 - e^{-\delta(1)s}\right)}{\delta(1)}, \quad I(s) = \beta + \frac{\left(1 - e^{-\delta(2)s}\right)}{\delta(2)}, \]

and setting \( \theta = 0 \) in (3.1), its corresponding joint survival probability is given by

\[
\mathbb{E}\left\{ e^{-\left(\Lambda_{t_2}^{(1)} - \Lambda_{t_1}^{(1)}\right)} e^{-\left(\Lambda_{t_2}^{(2)} - \Lambda_{t_1}^{(2)}\right)} \right\} = \exp\left[ \begin{array}{c} \frac{\alpha}{\delta(1)} \left(1 - e^{-\delta(1)(t_2-t_1)}\right) + \frac{\beta}{\delta(2)} \left(1 - e^{-\delta(2)(t_2-t_1)}\right) \end{array} \right] \times \exp\left[ \begin{array}{c} \frac{\alpha}{\delta(1)} \left(1 - e^{-\delta(1)(t_2-t_1)}\right) + \frac{\beta}{\delta(2)} \left(1 - e^{-\delta(2)(t_2-t_1)}\right) \end{array} \right] \times \exp\left[ \begin{array}{c} -\rho \int_{t_1}^{t_2} \left\{ \frac{\alpha}{\delta(1)} \left(1 - e^{-\delta(1)(t-t_1)}\right) + \frac{\beta}{\delta(2)} \left(1 - e^{-\delta(2)(t-t_1)}\right) \right\} ds \right]. \tag{3.2} \]

**Proof.** (3.1) follows if we set \( \gamma = 1 \) and \( \xi = 1 \) in (2.14) from which (3.2) follows if we set \( \theta = 0 \).

As we set \( \theta = 0 \) in (3.1) to get (3.2), event jump sizes in default intensities \( \{X_j^{(1)}\}_{j=1,2,\ldots} \) and \( \{X_j^{(2)}\}_{j=1,2,\ldots} \) are independent of each other. However they are still dependent each other as two jumps (i.e. two magnitudes of contribution of primary events to default intensities) \( X^{(1)} \) and \( X^{(2)} \) occur same time from sharing even jump frequency rate \( \rho \) even if their sizes are independent of each other.

**Corollary 3.2** The survival probability of firm 1 is given by

\[
\mathbb{E}\left\{ e^{-\left(\Lambda_{t_2}^{(1)} - \Lambda_{t_1}^{(1)}\right)} \right\} = \exp\left[ \begin{array}{c} \frac{\alpha e^{-\delta(1)(t_2-t_1)}}{\alpha + \frac{1}{\delta(1)} \left(1 - e^{-\delta(1)(t_2-t_1)}\right)} \end{array} \right] \times \exp\left[ \begin{array}{c} \frac{\beta e^{-\delta(2)(t_2-t_1)}}{\beta + \frac{1}{\delta(2)} \left(1 - e^{-\delta(2)(t_2-t_1)}\right)} \end{array} \right] \times \exp\left[ \begin{array}{c} -\rho \frac{\alpha e^{-\delta(1)(t_2-t_1)}}{\alpha + \frac{1}{\delta(1)} \left(1 - e^{-\delta(1)(t_2-t_1)}\right)} \end{array} \right] \left(1 - \frac{\alpha e^{-\delta(1)(t_2-t_1)}}{\alpha + \frac{1}{\delta(1)} \left(1 - e^{-\delta(1)(t_2-t_1)}\right)}\right)^{\alpha \rho} \delta_{\alpha+1} \tag{3.3} \right].

and the survival probability of firm 2 is given by
\[ \mathbb{E} \left\{ e^{-\left( \lambda_{12}^{(2)} - \lambda_{11}^{(2)} \right)} \right\} = \left\{ \frac{\beta e^{-\delta(t_2-t_1)}}{\beta + \frac{1}{\delta} \left( 1 - e^{-\delta(t_2-t_1)} \right)} \right\} \frac{\beta \rho}{\beta \rho + 1}, \tag{3.4} \]

which can be found in Dassios and Jang (2003).

**Proof.** If we set \( \gamma = 1 \) and \( \xi = 0 \) in (2.14), (3.3) follows. If we set \( \xi = 1 \) and \( \gamma = 0 \) in (2.14), (3.4) follows.

Due to the dependence of collateral even jumps of \( X^{(1)} \) and \( X^{(2)} \) with sharing event jump frequency rate \( \rho \), it shows that

\[ \mathbb{E} \left\{ e^{-\left( \lambda_{12}^{(1)} - \lambda_{11}^{(1)} \right)} e^{-\left( \lambda_{12}^{(2)} - \lambda_{11}^{(2)} \right)} \right\} \neq \mathbb{E} \left\{ e^{-\left( \lambda_{12}^{(1)} - \lambda_{11}^{(1)} \right)} \right\} \mathbb{E} \left\{ e^{-\left( \lambda_{12}^{(2)} - \lambda_{11}^{(2)} \right)} \right\}. \tag{3.5} \]

If event jump \( X^{(1)} \) occurs by a Poisson process \( M_t^{(1)} \) with its frequency rate \( \rho^{(1)} \) and event jump \( X^{(2)} \) occurs by a Poisson process \( N_t^{(2)} \) with its frequency \( \rho^{(2)} \) respectively and everything else is independent of each other, we have the joint survival probability of firm 1 and 2 at time \( t \) that is the product of the marginal survival probability of firm 1 and 2.

Now let us illustrate the calculations of the joint survival/default probabilities and relevant joint probabilities.

**Example 3.1**

We assume that the magnitude of the contribution to the default intensity of the firm 1 from the primary events is smaller than that of the firm 2. We also assume that the decay rate for the firm 1, that measures how quick the firm gets out of the influence of primary events lowering its default intensity rate, is higher than that for the firm 2. So the parameter values used to calculate the joint probabilities are

\[ \alpha = 10, \ \beta = 5, \ \delta^{(1)} = 0.5, \ \delta^{(2)} = 0.3, \ \rho = 4, \ \rho^{(1)} = 0 \text{ and } \rho^{(2)} = 1. \]

From the equation (3.1), (3.3) and (3.4), the calculations of the joint survival/default probabilities and relevant joint probabilities are shown in Table 3.1, 3.2, 3.3, and 3.4.

**Table 3.1.**

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \Pr(\tau_1 &gt; 1, \tau_2 &gt; 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.040875</td>
</tr>
<tr>
<td>0.5</td>
<td>0.040797</td>
</tr>
<tr>
<td>0</td>
<td>0.040720</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.040643</td>
</tr>
<tr>
<td>-1</td>
<td>0.040565</td>
</tr>
</tbody>
</table>

**Table 3.2.**

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \Pr(\tau_1 &gt; 1, \tau_2 \leq 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.42322</td>
</tr>
<tr>
<td>0.5</td>
<td>0.42330</td>
</tr>
<tr>
<td>0</td>
<td>0.42337</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.42345</td>
</tr>
<tr>
<td>-1</td>
<td>0.42353</td>
</tr>
</tbody>
</table>

**Table 3.3.**

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \Pr(\tau_1 \leq 1, \tau_2 &gt; 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.045414</td>
</tr>
<tr>
<td>0.5</td>
<td>0.045492</td>
</tr>
<tr>
<td>0</td>
<td>0.045570</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.045647</td>
</tr>
<tr>
<td>-1</td>
<td>0.045724</td>
</tr>
</tbody>
</table>

**Table 3.4.**

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \Pr(\tau_1 \leq 1, \tau_2 \leq 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.49049</td>
</tr>
<tr>
<td>0.5</td>
<td>0.49041</td>
</tr>
<tr>
<td>0</td>
<td>0.49034</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.49026</td>
</tr>
<tr>
<td>-1</td>
<td>0.49018</td>
</tr>
</tbody>
</table>
As we can see in (3.6), the survival probability of the firm 2 is very low. Hence the joint probabilities where the survivorship of the firm 2 is concerned are dominated by the survival probability of the firm 2 (see Table 3.1 and 3.3). On the contrary, the default probability of the firm 2 in (3.7) is very high. Hence joint probabilities where the defaultability of the firm 2 is concerned are dominated by the survival or default probability of the firm 1 (see Table 3.2 and 3.4).

Table 3.1 and 3.4 show that joint survival and default probability decrease as the value of copula parameter \( \theta \) becomes \(-1\) as time to default for each firm moves in the same direction. On the other hand, Table 3.2 and 3.3 show that joint probabilities increase as the value of copula parameter \( \theta \) becomes \(-1\) as time to default for each firm moves in the opposite direction. Hence when \( \theta = 1 \), we can consider applying the results to calculate joint survival and default probability for the firms in the complementary industry/sector. On the contrary, when \( \theta = -1 \), we can consider applying the results to calculate joint survival and default probability for the firms in the substitute industry/sector.

For an example when \( \theta = -1 \), let us consider that firm 1 produces cars run by petrol and firm 2 produces cars run by battery. If oil price surges that is an external event hits car manufacturing industry, consumers start swap petrol-running cars with battery-running cars. Hence these two firms are in substitute industry. For an example when \( \theta = 1 \), let us consider that firm 1 is an air-liner and firm 2 is a chain hotel. If a catastrophic event (e.g. September 11 WTC catastrophe) occurs, consumers travel less via air and subsequently hotel booking rates falls. Hence these two firms are in complementary industry.

Comparing joint default probability between complementary industry and substitute industry, joint default probability of the firms in complementary industry is higher than its counterpart in substitute industry that is economically intuitive (see Table 3.4). Secondly, comparing joint survival probability between complementary industry and substitute industry, joint survival probability of the firms in complementary industry is higher than its counterpart in substitute industry that is also economically intuitive (see Table 3.1). Lastly, relevant joint probabilities of the firms in substitute industry are higher than its counterpart in complementary industry as it is more likely that one firm would die (or survive) when the other firm survive (or die) if they are in substitute industry (see Table 3.2 and 3.3). It will be interesting to find these joint probabilities using other copulas with different marginal distributions.

Using Bayes’ rule, the conditional default probabilities between firm 1 and 2, denoted by \( p_{1|2} \) and \( p_{2|1} \) are given by

\[
p_{1|2} = \frac{p_{12}}{p_2} = \frac{\Pr(\tau_1 \leq t, \tau_2 \leq t)}{\Pr(\tau_2 \leq t)} = \frac{1 - \mathbb{E}\left\{e^{-\Lambda_i^{(1)}}\right\} - \mathbb{E}\left\{e^{-\Lambda_i^{(2)}}\right\} + \mathbb{E}\left\{e^{-\Lambda_i^{(1)}} e^{-\Lambda_i^{(2)}}\right\}}{1 - \mathbb{E}\left\{e^{-\Lambda_i^{(2)}}\right\}}
\]

and
Based on the same parameter values used in Example 3.1, let us illustrate the calculations of the conditional default probabilities.

**Example 3.2**

The calculation of the conditional default probabilities are shown in Table 3.5 and Table 3.6, respectively.

| $\theta$ | $p_{1|2}$ | $\theta$ | $p_{2|1}$ |
|----------|-----------|----------|-----------|
| 1        | 0.53682   | 1        | 0.91526   |
| 0.5      | 0.53673   | 0.5      | 0.91511   |
| 0        | 0.53665   | 0        | 0.91497   |
| -0.5     | 0.53656   | -0.5     | 0.91482   |
| -1       | 0.53648   | -1       | 0.91468   |

The conditional default probabilities in the tables above are rescaled to the joint default probabilities in Table 3.4 by the default probability of the firm 1 and 2 in (3.2), respectively. Hence the default probabilities of the firm 1 (or 2) given that the firm 2 (or 1) defaults are very similar to the unconditional default probability of the firm 1 (or 2) in (3.2). Two tables show that the conditional default probabilities between the firms in *complementary* industry are higher than its counterpart in *substitute* industry as it is more likely that one firm would die when the other firm die if they are in complementary industry, that is economically intuitive.

Next example shows that conditional default probabilities up to 1 can be achieved by changing the values of $\beta$ (or $\alpha$) and $\delta^{(2)}$ (or $\delta^{(1)}$) in the model specified.

**Example 3.3**

Using the same parameter values used in Example 3.1, the calculations of conditional default probabilities of $p_{2|1}$ at each value of $\beta$ and $\delta^{(2)}$ are shown in Table 3.7 and 3.8 with $\theta = 1$ respectively.

| $\beta$ | $p_{2|1}$ | $\delta^{(2)}$ | $p_{2|1}$ |
|---------|-----------|----------------|-----------|
| 10      | 0.72357   | 0.5            | 0.77552   |
| 5       | 0.91526   | 0.3            | 0.91526   |
| 3       | 0.97995   | 0.2            | 0.97489   |
| 1       | 0.99993   | 0.1            | 0.99935   |
| 0.1     | 1.00000   | 0.01           | 1.00000   |

The bigger $\beta$ and the higher $\delta^{(2)}$ i.e. the bigger the magnitude of the contribution to the default intensity of the firm from the primary events and the slower the firm gets out of the influence of primary events, the conditional default probabilities are getting closer to 1.
4. Measuring credit default swaps (CDS) rate

In this Section, we apply Section 3 results in pricing of a financial product. For that purpose, we choose credit default swaps (CDS) as three parties are involved in this contract, i.e. a reference credit, a CDS buyer and a CDS seller. In order to calculate credit default swaps (CDS) rate, firstly we assume that deterministic instantaneous rate of interest \( r \) for a zero-coupon default-free bond. Then its price at time 0, paying 1 at time \( t \) is given by

\[
B(0, t) = e^{-rt},
\]

where \( B(0, t) \) denotes the price of a default-free zero-coupon bond.

Now let us denote the default intensity process of the CDS buyer and seller by \( \lambda_t^{(b)} \) and \( \lambda_t^{(s)} \), respectively. We also specify the default intensity process of the reference credit by \( \lambda_t^{(rc)} \). For simplicity, we assume a deterministic recovery rate \( \pi \). Then credit default swaps (CDS) rate denoted by \( \pi \), that is adopted from Schönbucher (2003) is given by

\[
\pi = (1 - \pi) \frac{\sum_{k=1}^{k_N} e^{\pi_{rc,s}(0, t_{k-1}, t_k)} N \sum_{n=1}^{N} (t_{kn+1} - t_{kn}) B^b(0, t_{kn})}{\sum_{n=1}^{N} (t_{kn+1} - t_{kn}) B^b(0, t_{kn})},
\]

where

\[
e^{\pi_{rc,s}(0, t_{k-1}, t_k)} = \mathbb{E} \left[ \exp \left( - \int_0^{t_k} r_s ds \right) \exp \left( \int_0^{t_{k-1}} \lambda_s^{(rc)} ds \right) \exp \left( - \int_0^{t_k} \lambda_s^{(s)} ds \right) \right],
\]

\[
B^b(0, t_{kn}) = \mathbb{E} \left[ \exp \left( - \int_0^{t_{kn}} (r_s + \lambda_s^{(b)}) ds \right) \right],
\]

and \( t_{k_1} < t_{k_2} < \cdots < t_{k_N} \). Assuming that \( r_t \) and \( \lambda_t^{(i)} \) are independent of each other and that \( \lambda_t^{(i)} \) is stationary, equation (4.3) and (4.4) can be expressed as

\[
e^{\pi_{rc,s}(0, t_{k-1}, t_k)} = e^{-r t_k} \times \mathbb{E} \left[ e^{-\Lambda_{t_k-1}^{(rc)} - \Lambda_{t_k-1}^{(s)}} \mathbb{E} \left[ \exp \left( - \int_{t_{k-1}}^{t_k} \lambda_s^{(s)} ds \right) \right] - \mathbb{E} \left[ e^{-\Lambda_{t_k}^{(rc)} - \Lambda_{t_k}^{(s)}} \right] \right]
\]

\[
e^{\pi_{rc,s}(0, t_{k-1}, t_k)} = e^{-r t_k} \times \mathbb{E} \left[ e^{-\Lambda_{t_k-1}^{(rc)} - \Lambda_{t_k-1}^{(s)}} \mathbb{E} \left[ e^{-\Lambda_{t_k}^{(s)} - \Lambda_{t_k-1}^{(s)}} \right] - \mathbb{E} \left[ e^{-\Lambda_{t_k}^{(rc)} - \Lambda_{t_k}^{(s)}} \right] \right]
\]

and

\[
B^b(0, t_{kn}) = e^{-r t_{kn}} \mathbb{E} \left( e^{-\Lambda_{t_{kn}}^{(k)}} \right).
\]
Using equation (4.6), we may consider pricing defaultable bonds as well as credit spread between default-free bond and defaultable bond.

Let us illustrate the calculations of credit default swaps (CDS) rates using the expressions derived above.

**Example 4.1**
The parameter values used to calculate (4.2) are

\[ r = 0.05, \quad \pi = 50\%, \quad N = 2, \quad t_{k_0} = 0, \quad t_{k_1} = 0.5, \quad t_{k_2} = 1. \]

We assume that the default intensity processes of CDS buyer and seller follows \( \lambda_t^{(1)} \), i.e.

\[ \lambda_t^{(1)} = \lambda_t^{(b)} = \lambda_t^{(s)} \]

and that the default intensity process of reference credit follows \( \lambda_t^{(2)} \), i.e.

\[ \lambda_t^{(2)} = \lambda_t^{(rc)}. \]

Using the same parameter values as in Example 3.1 for \( \lambda_t^{(i)} \), the calculations of credit default swaps (CDS) rates are shown in Table 4.1.

**Table 4.1.**

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \bar{\pi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3656.2bp</td>
</tr>
<tr>
<td>0.5</td>
<td>3656.8bp</td>
</tr>
<tr>
<td>0</td>
<td>3657.5bp</td>
</tr>
<tr>
<td>-0.5</td>
<td>3658.2bp</td>
</tr>
<tr>
<td>-1</td>
<td>3658.8bp</td>
</tr>
</tbody>
</table>

Table 4.1 shows that the CDS rate is getting higher as \( \theta \) moves to -1. When \( \theta = 1 \), joint default probability between the reference credit and the CDS seller is higher than its counterpart when \( \theta = -1 \) (see Table 3.4). In other words, when \( \theta = -1 \), there are less chance that the CDS seller default when the reference credit defaults, e.g. when a sub-prime mortgage holder who is the reference credit defaults, it is less likely an investment banker who is the CDS seller defaults compared to when \( \theta = 1 \). Therefore the CDS buyer needs to pay higher CDS rate/fee as \( \theta \) moves to -1. It is of interest to use other copulas in modelling dependence between the vector of jumps \( \left( X_j^{(1)}, X_j^{(2)}, \cdots, X_j^{(n)} \right) \) and calculate CDS rates via numerical methods.

Assuming that \( \theta = 1 \), we now examine the effect on credit default swaps (CDS) rate caused by changes in the value of \( \beta \) and \( \delta^{(rc)} \) for the reference credit and by changes in the value of \( \alpha^{(s)} \) and \( \delta^{(s)} \) for the CDS seller.

**Example 4.2**
Using the same parameter values used in Example 4.1, the calculations of credit default swaps (CDS) rates caused by changes in the value of \( \beta \) and \( \delta^{(rc)} \) for the reference credit and by changes in the value of \( \alpha^{(s)} \) and \( \delta^{(s)} \) for the CDS seller are shown in Table 4.2 and Table 4.3, respectively.
Compared to CDS rates in Table 4.1, we can see more clear relationship between CDS rates and the parameter values of default intensity for the reference credit and for the CDS seller in Table 4.2 and 4.3, respectively. In Table 4.2, we can see that CDS rate is converging by lowering the values of $\beta$ and $\delta^{(rc)}$ for the reference credit, respectively as the CDS seller’s default intensity is not as bad as its counterpart for the reference credit. On the contrary, Table 4.3 shows that CDS rate is getting lower to 0 by decreasing the value of $\alpha^{(s)}$ and $\delta^{(s)}$ for the CDS seller. From the CDS buyers’ point of view, it is better for them to purchase a CDS contract that the CDS seller is less likely to default. As long as the CDS seller’s credit is strong enough, they can hedge against the default risk of the reference credit using a CDS contract. Hence the lower the CDS rate is, the more likely the CDS seller defaults. The worst case scenario for the CDS buyer is when both the reference credit and the CDS seller default.

5. Conclusion

For collateral default intensity, we used multivariate shot noise intensity process where jumps (i.e. magnitude of contribution of primary events to default intensities) occur simultaneously and their sizes are correlated. To count collateral event jumps in default intensities, a homogeneous Poisson process was used and to model correlations between event jump sizes, we used Farlie-Gumbel-Morgenstern (FGM) copulas with exponential margins. Applying copula-dependent collateral default intensity to multivariate Cox process, joint survival/default probability and other relevant joint probabilities were derived via the joint Laplace transform. Standard martingale theory was used to derive the joint Laplace transforms. As an example of using joint survival/default probability, we showed how it can be used to calculate credit default swaps (CDS) rate. We also examined the sensitivity of CDS rates changing the parameter values of collateral default intensity between the reference credit and the CDS seller. We leave pricing of credit portfolio derivatives such as CDOs for further research based on our modeling approach where related firms are in the complementary or substitute industry/sector.

References


