Double Shot Noise Process and Its Application
In Insurance

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Double shot noise process and its application in insurance

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Abstract

We consider a compound Cox model of insurance risk with the additional economic assumption of a positive interest rate. To accommodate both stochastic claim intensity and the time value of claims within the aggregate loss, we use a double shot noise process. Using its generator, we derive the moments of aggregate accumulated/discounted claims where the claim arrival process follows a Cox process with shot noise intensity. Removing the parameters in a double shot noise process gradually, we show that it becomes a compound Cox process with shot noise intensity, a single shot noise process and a compound Poisson process, respectively. Numerical comparisons are shown between the moments (i.e. means and variances) of a compound Poisson model and their counterparts of a compound Cox model with/without considering a positive interest rate. For that purpose, we assume that claim sizes and primary event sizes follow an exponential distribution respectively.

Keywords: Double shot noise process; a Cox process; stochastic intensity and time value of claims; aggregate accumulated/discounted claims.

1. Introduction

In insurance modelling, the assumption that resulting claims occur in terms of a Poisson process is inadequate as it has deterministic intensity. To accommodate stochastic nature of the claim arrivals from flood, windstorm, hail, bushfire and earthquake in practice, an alternative point process needs to be used to generate their arrivals. Dassios and Jang (2003) used a Cox process with shot noise intensity to price catastrophe reinsurance and derivatives. Albrecher and Asmussen (2006) applied shot noise Cox processes on the probability of ruin.

In classical risk theory, we often implicitly assume that interest rates are zero within the claims process. Delbaen and Haegendorf (1987) extended the classical risk theory to consider the effect of the introduction of interest rate factors, leading to an explosion of literature in this subject (Wilmot 1989; Paulsen 1998; Léveillé and Garrido 2001; Jang 2004 and Kim and Kim 2007).

So far however no papers accommodate both a positive interest rate to claims and the stochastic nature of claim frequency within the claim arrival process. To achieve these, we introduce a specific model that has the following structure:

\[ L_t = \sum_{i=1}^{N_t} X_i e^{\delta(t-S_i)} , \tag{1.1} \]

where \( L_t \) is the aggregate accumulated claim amounts up to time \( t \), \( X_i \), \( i = 1, 2, \cdots \), are the claim amounts which are assumed to be independent and identically distributed with distribution function \( H(x) \) \( (x > 0) \), \( S_i \) is the time of claim \( i \), \( \delta \) is the risk-free force of interest rate and to deal with stochastic nature of catastrophic loss arrival in practice, we use a Cox process for \( N_t \).

The Cox process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Therefore the Cox process can be viewed as a two step
randomisation procedure. A process $\lambda_t$ is used to generate another process $N_t$ by acting its intensity. That is, $N_t$ is a Poisson process conditional on $\lambda_t$ which itself is a stochastic process.

Losses arising from a catastrophe depend on its intensity. One of the processes that can be used to measure the impact of catastrophic events is the shot noise process. Previous works of insurance application using shot noise process and a Cox process with shot noise intensity can be found in Klüppelberg & Mikosch (1995), Dassios & Jang (2003, 2005 and 2008) and Jang & Krvavych (2004). Jang and Fu (2008) also used a Cox process with shot noise intensity to model operational risk. The shot noise process is particularly useful in loss arrival process as it measures the frequency, magnitude and time period needed to determine the effect of catastrophic events. As time passes, the shot noise process decreases as more and more losses are settled. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of a Cox process to measure the number of catastrophic losses, i.e. we will use it as an intensity function to generate a Cox process. We will adopt the shot noise process used by Cox & Isham (1980):

$$\lambda_t = \lambda_0 e^{-\kappa t} + \sum_{j=1}^{M_t} Y_j e^{-\kappa(t-U_j)},$$

where:

- $\lambda_0$ is the initial value of $\lambda_t$ that is carried on from catastrophic events incurred previously;
- $\{Y_j\}_{j=1,2,...}$ is a sequence of independent and identically distributed random variables with distribution function $G(y)$ ($y > 0$) and $E(Y) = \mu_1$ (i.e. magnitude of contribution of catastrophic event $j$ to intensity);
- $\{U_j\}_{j=1,2,...}$ is the sequence representing the event times of a Poisson process $M_t$ with constant intensity $\rho$;
- $\kappa$ is the rate of exponential decay.

Catastrophic events may take long to materialise so the decay rate may not be exponential. It is assumed to be of this form for a matter of convenience, i.e. closed-form expressions of final results are easily derived. We also make the additional assumption that a Poisson process $M_t$ and the sequences $\{X_i\}_{i=1,2,...}$ and $\{Y_i\}_{j=1,2,...}$ are independent of each other.

The above two equations can be written in terms of stochastic differential equation (SDE), i.e.

$$dL_t = \delta L_t dt + dV_t,$$  

where

$$V_t = \sum_{i=1}^{N_t} X_i$$

and

$$d\lambda_t = -\kappa \lambda_t dt + dC_t,$$  

where

$$C_t = \sum_{j=1}^{M_t} Y_j.$$

If we replace $\delta$ with $-\delta$ in (1.3), it becomes
\[ d\xi_t = -\delta \xi_t dt + dV_t \quad (1.5) \]

and together with

\[ d\lambda_t = -\kappa \lambda_t dt + dC_t, \quad (1.6) \]

we have a double shot noise process. For details of double shot noise process, we refer Dassios (1987).

With the above model specification and assuming that \( L_0 = 0 \) and \( \lambda_t \) is stationary, in Section 2 we derive the moments (i.e. expectation and variance) of \( L_t \) and of \( L_0^t = e^{-\delta t} L_t \), where

\[ L_0^t = \sum_{i=1}^{N_t} X_i e^{-\delta S_i} \quad (1.7) \]

is the aggregate discounted claim amounts up to time \( t \). To do so, we use the generator of the process \( (\xi_t, \lambda_t, t) \). In Section 3 and Section 4, deleting the parameters in a double shot noise process, we show that it becomes a compound Cox process with shot noise intensity, a single shot noise process, a compound Poisson process and derive their moments respectively. Assuming that claim sizes and primary event sizes follow an exponential distribution respectively, we obtain the explicit expressions of these moments and show their numerical calculations. Section 5 contains some concluding remarks.

2. Double shot noise process and its generator

The generator of the process \( (\xi_t, \lambda_t, t) \) acting on a function \( f(\xi, \lambda, t) \) belonging to its domain is given by

\[
\mathbf{A} f(\xi, \lambda, t) = \frac{\partial f}{\partial t} - \delta \xi \frac{\partial f}{\partial \xi} + \lambda \left[ \int_0^\infty f(\xi + x, \lambda, t) \, dH(x) - f(\xi, \lambda, t) \right] + \rho \left[ \int_0^\infty f(\xi, \lambda + y, t) \, dG(y) - f(\xi, \lambda, t) \right], \quad (2.1)
\]

where \( f : (0, \infty) \times (0, \infty) \times \mathbb{R}^+ \to (0, \infty) \). It is sufficient that \( f(\xi, \lambda, t) \) is differentiable w.r.t. \( \xi, \lambda, t \) for all \( \xi, \lambda, t \) and that

\[
\left| \int_0^\infty f(\cdot, \xi + x, \cdot) \, dH(x) - f(\cdot, \xi, \cdot) \right| < \infty
\]

and

\[
\left| \int_0^\infty f(\cdot, \lambda + y, \cdot) \, dG(y) - f(\cdot, \lambda, \cdot) \right| < \infty
\]

for \( f(\xi, \lambda, t) \) to belong to the domain of the generator \( \mathbf{A} \). For details of finding the generator of the process \( (\xi_t, \lambda_t, t) \) applying the piecewise deterministic Markov processes (PDMPs) theory, we refer Dassios and Jang (2008).

2.1 Expectation of the aggregate discounted claim
In this section, assuming that $L_0 = 0$ and $\lambda_t$ is stationary, we examine the mean of the aggregate discounted claim. To do so, let us begin with deriving the expectation of $\xi_t$ at time $t$ assuming that $\xi_0$ is given. If we set $f(\xi, \lambda, t) = f(\xi) = \xi$ in (2.1), then we have

$$A \xi = -\delta \xi + m_1 \lambda,$$

where

$$m_1 = \int_0^\infty x dH(x) < \infty.$$

From the Dynkin’s formula conditioning on $\lambda_t$, we have

$$E(\xi_t \mid \xi_0, \lambda_t) = \xi_0 - \delta \int_0^t E(\xi_s \mid \xi_0, \lambda_s) ds + m_1 \int_0^t \lambda_s ds.$$

Differentiating (2.2) w.r.t. $t$, we have

$$\frac{dE(\xi_t)}{dt} = -\delta E(\xi_t \mid L_0, \lambda_t) + m_1 \lambda_t$$

and solving this differential equation, we obtain

$$E(\xi_t \mid \xi_0, \lambda_t) = \xi_0 e^{-\delta t} + m_1 e^{-\delta t} \int_0^t e^{\delta s} \lambda_s ds.$$

Now take conditional expectation on $\lambda_t$ in (2.3) and assume that $\lambda_0$ is given, then we have

$$E(\xi_t \mid \xi_0, \lambda_0) = \xi_0 e^{-\delta t} + m_1 e^{-\delta t} \int_0^t e^{\delta s} E(\lambda_s \mid \lambda_0) ds.$$

It is known in Jang and Krvavych (2004) that the expectation of $\lambda_t$ given $\lambda_0$ is given by

$$E(\lambda_t \mid \lambda_0) = \lambda_0 e^{-\kappa t} + \mu_1 \rho \left( \frac{1 - e^{-\kappa t}}{\kappa} \right)$$

and that if $\lambda_t$ is stationary (i.e. let $t \to \infty$), it becomes

$$\frac{\mu_1 \rho}{\kappa},$$

where $\mu_1 = \int_0^\infty y dG(y) < \infty$.

Setting (2.5) in (2.4), it is given that

$$E(\xi_t \mid \xi_0, \lambda_0) = \xi_0 e^{-\delta t} + m_1 \left\{ \frac{\mu_1 \rho}{\kappa} \left( \frac{1 - e^{-\delta t}}{\delta} \right) + \left( \lambda_0 - \frac{\mu_1 \rho}{\kappa} \right) \left( \frac{e^{-\kappa t} - e^{-\delta t}}{\delta - \kappa} \right) \right\},$$

where $\delta \neq \kappa$ and if $\lambda_t$ is stationary (i.e. let $t \to \infty$), it becomes

$$\xi_0 e^{-\delta t} + m_1 \frac{\mu_1 \rho}{\kappa} \left( \frac{1 - e^{-\delta t}}{\delta} \right)$$

Replace $-\delta$ with $\delta$ in (2.7) and (2.8) respectively and for simplicity, let us assume that $L_0 = 0$. Then we have

$$E(L_t \mid \lambda_0) = m_1 \left\{ \frac{\mu_1 \rho}{\kappa} \left( \frac{e^{\delta t} - 1}{\delta} \right) + \left( \lambda_0 - \frac{\mu_1 \rho}{\kappa} \right) \left( \frac{e^{\delta t} - e^{-\kappa t}}{\delta + \kappa} \right) \right\},$$

(2.9)
and if \( \lambda_t \) is stationary (i.e. let \( t \rightarrow \infty \)), it becomes

\[
m_1 \frac{\mu_1 \rho}{\kappa} \left( \frac{e^{\delta t} - 1}{\delta} \right) = m_1 \frac{\mu_1 \rho}{\kappa} \hat{s}_1 \big|_{\delta} \tag{2.10}
\]

Multiplying \( e^{-\delta t} \) both sides in (2.9) and (2.10) respectively, we have the mean of the aggregate discounted claim given \( \lambda_0 \),

\[
E(L_t^0 | \lambda_0) = m_1 \left\{ \frac{\mu_1 \rho}{\kappa} \left( \frac{1 - e^{-\delta t}}{\delta} \right) + \left( \lambda_0 - \frac{\mu_1 \rho}{\kappa} \right) \left( \frac{1 - e^{-\left(\delta + \kappa\right)t}}{\delta + \kappa} \right) \right\} \tag{2.11}
\]

and if \( \lambda_t \) is stationary (i.e. let \( t \rightarrow \infty \)), it becomes

\[
m_1 \frac{\mu_1 \rho}{\kappa} \left( \frac{1 - e^{-\delta t}}{\delta} \right) = m_1 \frac{\mu_1 \rho}{\kappa} \hat{s}_1 \big|_{\delta} \tag{2.12}
\]

2.2 Variance of the aggregate discounted claim

Similar to Section 2.1, assuming that \( L_0 = 0 \) and \( \lambda_t \) is stationary, we examine the variance of the aggregate discounted claim. To do so, let us start with deriving the second moment of \( \xi_t \) at time \( t \) assuming that \( \xi_0 \) and \( \lambda_0 \) are given.

Set \( f(\xi, \lambda, t) = f(\xi) = \xi^2 \) in (2.1), then we can easily obtain

\[
E(\xi_t^2 | \xi_0, \lambda_0) = \xi_0^2 e^{-2\delta t} + 2m_1 e^{-2\delta t} \int_0^t e^{2\delta s} E(\xi_s \lambda_s | \xi_0, \lambda_0) ds + m_2 e^{-2\delta t} \int_0^t e^{2\delta s} E(\lambda_s | \lambda_0) ds, \tag{2.13}
\]

where

\[
m_2 = \int_0^t x^2 dH(x) < \infty.
\]

To obtain the expression of (2.13), firstly we need to find the expression for \( E(\xi_s \lambda_s | \xi_0, \lambda_0) \). So setting \( f(\xi, \lambda, t) = f(\xi, \lambda) = \xi \lambda \) in (2.1), we have

\[
E(\xi_t \lambda_t | \xi_0, \lambda_0) = \xi_0 \lambda_0 e^{-\left(\delta + \kappa\right)t} + m_1 e^{-\left(\delta + \kappa\right)t} \int_0^t e^{(\delta + \kappa)s} E(\lambda_s^2 | \lambda_0) ds + \mu_1 \rho e^{-\left(\delta + \kappa\right)t} \int_0^t e^{(\delta + \kappa)s} E(\xi_s | \xi_0, \lambda_0) ds. \tag{2.14}
\]

It is known in Jang and Krvavych (2004) that

\[
E(\lambda_t^2 | \lambda_0) = \lambda_0^2 e^{-2\kappa t} + 2\mu_1 \rho \left( \lambda_0 - \frac{\mu_1 \rho}{\kappa} \right) \left( e^{-\kappa t} - e^{-2\kappa t} \right) + \left( \frac{\mu_1 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) (1 - e^{-2\kappa t}). \tag{2.15}
\]

and that if \( \lambda_t \) is stationary (i.e. let \( t \rightarrow \infty \)), it becomes
where \( \mu_2 = \int_0^\infty y^2 dG(y) < \infty \). Hence using (2.15) and (2.7), (2.14) is given that

\[
E(\xi_t, \lambda_t) = \xi_0 \lambda_0 e^{-(\delta + \kappa)t} + m_1 \left\{ \xi_0 - m_1 \frac{\mu_1 \rho}{\kappa} \right\} + m_1 \frac{\mu_1 \rho}{\kappa} \left( \frac{e^{-\delta t} - e^{-(\delta + \kappa)t}}{\delta - \kappa} \right).
\]

Assuming that \( \xi_0 = 0 \) and that \( \lambda_t \) is stationary for (2.17), it becomes

\[
\left\{ m_1 \left( \frac{\mu_1^2 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) + m_1 \frac{\mu_1^2 \rho^2}{\kappa} \right\} \left( \frac{1 - e^{-(\delta + \kappa)t}}{\delta + \kappa} \right) - m_1 \frac{\mu_1^2 \rho^2}{\delta \kappa} \left( \frac{e^{-\delta t} - e^{-(\delta + \kappa)t}}{\kappa} \right).
\]

Replace \(-\delta\) with \(\delta\) in (2.18), then we have

\[
\left\{ m_1 \left( \frac{\mu_1^2 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) - m_1 \frac{\mu_1^2 \rho^2}{\delta \kappa} \right\} \left( \frac{e^{(\delta - \kappa)t} - 1}{\delta - \kappa} \right) + m_1 \frac{\mu_1^2 \rho^2}{\delta \kappa} \left( \frac{e^{\delta t} - e^{(\delta - \kappa)t}}{\kappa} \right)
\]

and multiply \(e^{-\delta t}\) both sides, the joint expectation of \(L^0_t\) and \(\lambda_t\) at time \(t\) is given by

\[
\left\{ m_1 \left( \frac{\mu_1^2 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) - m_1 \frac{\mu_1^2 \rho^2}{\delta \kappa} \right\} \left( \frac{e^{-\delta t} - e^{-\delta t}}{\delta - \kappa} \right) + m_1 \frac{\mu_1^2 \rho^2}{\delta \kappa} \left( \frac{1 - e^{-\delta t}}{\kappa} \right).
\]

Now setting (2.17) and (2.5) into (2.13), we have
\[ E(\xi_t^2 \mid \xi_0, \lambda_0) = \xi_0^2 e^{-2\delta t} \]

\[
\begin{bmatrix}
- \frac{m_1}{\delta - \kappa} \left\{ \lambda_0^2 - \frac{2m_1 \mu_1 \rho}{\kappa} (\lambda_0 - \frac{\mu_1 \rho}{\kappa}) - \left( \frac{\mu_1 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) \right\} \\
\frac{m_1}{\delta - \kappa} \left\{ \lambda_0 - \frac{m_1 \mu_1 \rho}{\kappa} - \left( \frac{\mu_1 \rho}{\kappa} \right) \left( \frac{1}{\delta - \kappa} \right) \right\} \\
\frac{m_1}{\delta - \kappa} \left\{ \lambda_0 - \frac{m_1 \mu_1 \rho}{\kappa} - \left( \frac{\mu_1 \rho}{\kappa} \right) \left( \frac{1}{\delta - \kappa} \right) \right\}
\end{bmatrix}
\left( \frac{e^{-(\delta + \kappa)t}}{\delta - \kappa} \right)
\]

We firstly assume that \( \xi_0 = 0 \) in \( \text{Var}(\xi_t \mid \xi_0, \lambda_0) = E(\xi_t^2 \mid \xi_0, \lambda_0) - \{ E(\xi_t \mid \xi_0, \lambda_0) \}^2 \) and that \( \lambda_t \) is stationary. Then it is given by

\[
\begin{align*}
2m_1 & \left\{ - \frac{\mu_1 \rho}{\kappa} \left( \frac{m_1}{\delta} \right) \right\} - \left\{ m_1 \left( \frac{\mu_1 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) + \frac{m_1 \mu_1^2 \rho^2}{\delta} \right\} \left( \frac{1}{\delta + \kappa} \right) \left( \frac{e^{-(\delta + \kappa)t}}{\delta - \kappa} \right) \\
-2m_1 & \left\{ - \frac{\mu_1 \rho}{\kappa} \left( \frac{m_1}{\delta} \right) \right\} - \left\{ m_1 \left( \frac{\mu_1 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) + \frac{m_1 \mu_1^2 \rho^2}{\delta} \right\} \left( \frac{1}{\delta + \kappa} \right) \left( \frac{e^{-2\delta t}}{\delta - \kappa} \right) \\
+2m_1 & \left\{ - \frac{\mu_1 \rho}{\kappa} \left( \frac{m_1}{\delta} \right) \right\} \left( \frac{e^{-\delta t} - e^{-2\delta t}}{\delta} \right) \\
+ \left[ 2m_1 \left\{ \frac{\mu_1^2 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right\} + \frac{m_1 \mu_1^2 \rho^2}{\delta} \right] \left( \frac{1}{\delta} \right) \left( 1 - e^{-2\delta t} \right) \\
- \left\{ m_1 \left\{ \frac{\mu_1 \rho}{\kappa} \left( \frac{1}{\delta} \right) \right\} \right\}^2
\end{align*}
\]

Replacing \(-\delta \) with \( \delta \) in (2.22), we have
and multiply with \( e^{-2\delta t} \) both sides, then the variance of the discounted aggregate claims, where \( \lambda_t \) is stationary, is given by

\[
-2m_1 \left[ \frac{\mu_1 \rho}{\kappa} \left\{ \frac{m_1 \mu_1 \rho^2}{\delta \kappa} \right\} + \left\{ m_1 \left( \frac{\mu_1^2 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) - \frac{m_1 \mu_1^2 \rho^2}{\delta \kappa} \right\} \frac{1}{\delta - \kappa} \right] \left( \frac{e^{(\delta - \kappa)t}}{\delta + \kappa} \right) \\
+2m_1 \left[ -\frac{\mu_1 \rho}{\kappa} \left\{ \frac{m_1 \mu_1 \rho^2}{\delta \kappa} \right\} + \left\{ m_1 \left( \frac{\mu_1^2 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) - \frac{m_1 \mu_1^2 \rho^2}{\delta \kappa} \right\} \frac{1}{\delta - \kappa} \right] \left( \frac{e^{2\delta t}}{\delta + \kappa} \right) \\
-2m_1 \frac{\mu_1 \rho}{\kappa} \left\{ \frac{m_1 \mu_1 \rho^2}{\delta \kappa} \right\} \left( \frac{e^{\delta t} - e^{2\delta t}}{\delta} \right) \\
+ \left[ -\frac{2m_1}{\delta - \kappa} \left\{ \frac{m_1 \mu_1 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right\} - \frac{m_1 \mu_1 \rho^2}{\delta \kappa} \right] \left( \frac{e^{2\delta t} - 1}{2\delta} \right) \\
- \left[ m_1 \left\{ \frac{\mu_1 \rho}{\kappa} \left( \frac{e^{\delta t} - 1}{\delta} \right) \right\} \right]^2 
\]  

\tag{2.23}

and that if \( \lambda_t \) is stationary (i.e. let \( t \to \infty \)), it becomes

\[
2m_1 \left[ \left\{ m_1 \left( \frac{\mu_1^2 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) \right\} - \frac{m_1 \mu_1 \rho^2}{\delta \kappa^2} \right] \left( \frac{1}{\delta + \kappa} \right) \\
+2m_1 \frac{m_1 \mu_1 \rho^2}{\delta \kappa^2} \left( \frac{1 - e^{-\delta t}}{\delta} \right) \\
+ \left[ m_2 \frac{\mu_1 \rho}{\kappa} - \frac{2m_1}{\delta - \kappa} \left\{ m_1 \left( \frac{\mu_1^2 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) \right\} \right] \frac{1 - e^{-2\delta t}}{2\delta} \\
-2m_1 \left\{ m_1 \left( \frac{\mu_1^2 \rho^2}{\kappa^2} + \frac{\mu_2 \rho}{2\kappa} \right) - \frac{m_1 \mu_1 \rho^2}{\delta \kappa} \right\} \frac{1}{\delta - \kappa} \left( \frac{e^{-(\delta + \kappa)t}}{\delta + \kappa} \right) \\
- \left\{ m_1 \left( \frac{\mu_1 \rho}{\kappa} \left( \frac{1 - e^{-\delta t}}{\delta} \right) \right) \right\}^2 
\]  

\tag{2.24}

It is known in Jang (1998) that the variance of \( \lambda_t \) given \( \lambda_0 \) is given by

\[
\text{Var}(\lambda_t | \lambda_0) = \mu_2 \rho \left( \frac{1 - e^{-2\delta t}}{2\kappa} \right) 
\]  

\tag{2.25}

and that if \( \lambda_t \) is stationary (i.e. let \( t \to \infty \)), it becomes

\[
\frac{\mu_2 \rho}{2\kappa}.
\]

\section*{3. Variations from the mean of double shot noise process}

Set \( \delta = 0 \) in (1.5), then we have a compound Cox process with shot noise intensity as aggregate claim process. Putting \( \delta = 0 \) either in (2.10) or in (2.12), its mean is given by

\[
m_1 \frac{\mu_1 \rho}{\kappa} \frac{t}{\kappa} 
\]  

\tag{3.1}

as the time value of claims is no longer considered. The same result can also be found in Dassios and Jang (2003).

Replace \( -\kappa \) with \( \kappa \) in (1.6) and for simplicity, let us assume that \( \lambda_0 = 0 \), then (2.5) becomes

\[
\mu_1 \rho \left( \frac{e^{\delta t} - 1}{\kappa} \right) = \mu_1 \rho \frac{e^{\delta t}}{\kappa} \text{ at } \kappa.
\]  

\tag{3.2}
Multiply \( e^{-\kappa t} \) in (3.2) then it becomes

\[
\mu_1 \beta \left( \frac{1 - e^{-\kappa t}}{\kappa} \right) = \mu_1 \rho \tilde{a} \tilde{a}_{\delta t} \text{ at } \kappa. \tag{3.3}
\]

Considering \( \rho \) deterministic claim intensity, \( \mu_1 \) the mean of claim sizes and \( \kappa \) the risk-free force of interest rate, then (3.2) and (3.3) are the means of aggregate accumulated/discounted claims respectively where the claim arrival process follows a Poisson process. The same result can also be found in Jang (2004).

Set \( \kappa = 0 \) either in (3.2) or in (3.3), then it is given by

\[
\mu_1 \rho t \tag{3.4}
\]
as the time value of claims is no longer considered. This is the mean of aggregate claims where the claim arrival process follows a Poisson process.

For the numerical comparison, let us denote \( \rho = \lambda \), \( \mu_1 = m_1 \) and \( \kappa = \delta \) in (3.2)-(3.4). Using an exponential distribution of \( G(y) \) and \( H(x) \) for event jump and claim size respectively, i.e.

\[
g(y) = \alpha e^{-\alpha y} \quad \text{and} \quad h(x) = \beta e^{-\beta x} \quad \text{with} \quad \alpha > 0, \ \beta > 0,
\]

(2.12), (3.1), (3.3) and (3.4) are given by

\[
\frac{1}{\beta} \frac{\rho}{\alpha \kappa} \tilde{a} \tilde{a}_{\delta t}, \quad \text{(Shot noise Cox with positive interest)} \tag{3.5}
\]

\[
\frac{1}{\beta} \frac{\rho}{\alpha \kappa} t, \quad \text{(Shot noise Cox with no interest)} \tag{3.6}
\]

\[
\frac{1}{\beta} \lambda \tilde{a} \tilde{a}_{\delta t}, \quad \text{(Poisson with positive interest)} \tag{3.7}
\]

\[
\frac{1}{\beta} \lambda t. \quad \text{(Poisson with no interest)} \tag{3.8}
\]

If we compare between (3.5) and (3.7) (or between (3.6) and (3.8)), we can easily notice a difference that arises due to stochastic claim intensity (i.e. the stationary mean of shot noise intensity \( \lambda \), of a Cox process, \( \frac{\rho}{\alpha \kappa} \)) and deterministic claim intensity (i.e. the mean of a Poisson process, \( \lambda \)). Obviously, if \( \lambda = \frac{\rho}{\alpha \kappa} \), (3.5) and (3.7) (or (3.6) and (3.8)) have the same values.

Now let us illustrate the calculations of the means of a compound Poisson model and their counterpart of a compound Cox model with/without considering a positive interest rate.

**Example 3.1**

The parameter values used to calculate the means are

\[
\beta = 0.01, \ \lambda = 50, \ \delta = 0.05, \ t = 1
\]

and

\[
\alpha = 0.5, \ \rho = 4, \ \kappa = 0.1.
\]

The calculations of the means of a compound Poisson model and their counterpart of a compound Cox model with/without considering a positive interest rate are shown in Table 3.1.
Table 3.1.

<table>
<thead>
<tr>
<th>Shot noise Cox with positive interest</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shot noise Cox with no interest</td>
<td>7803.3</td>
</tr>
<tr>
<td>Poisson with positive interest</td>
<td>8000.0</td>
</tr>
<tr>
<td>Poisson with no interest</td>
<td>4877.1</td>
</tr>
</tbody>
</table>

The next example shows the effect on the means of Cox with positive interest case caused by changes in the value of $\alpha$, $\rho$ and $\kappa$ respectively.

**Example 3.2**
Using the same parameter values in Example 3.1, the calculations of the means of Cox with positive interest case caused by changes in the values of $\alpha$, $\rho$ and $\kappa$ are shown in Table 3.2-3.4.

<table>
<thead>
<tr>
<th>Table 3.2.</th>
<th>Table 3.3.</th>
<th>Table 3.4.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.01$</td>
<td>$\rho = 2$</td>
<td>$\kappa = 0.01$</td>
</tr>
<tr>
<td>390,160</td>
<td>3,901.6</td>
<td>78,033</td>
</tr>
<tr>
<td>$\alpha = 0.1$</td>
<td>$\rho = 4$</td>
<td>$\kappa = 0.1$</td>
</tr>
<tr>
<td>39,016</td>
<td>7,803.3</td>
<td>7,803.3</td>
</tr>
<tr>
<td>$\alpha = 0.5$</td>
<td>$\rho = 10$</td>
<td>$\kappa = 0.5$</td>
</tr>
<tr>
<td>7,803.3</td>
<td>19,508</td>
<td>1,560.7</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>$\rho = 20$</td>
<td>$\kappa = 1$</td>
</tr>
<tr>
<td>3,901.6</td>
<td>39,016</td>
<td>780.33</td>
</tr>
<tr>
<td>$\alpha = 2$</td>
<td>$\rho = 50$</td>
<td>$\kappa = 2$</td>
</tr>
<tr>
<td>1,950.8</td>
<td>97,541</td>
<td>390.16</td>
</tr>
</tbody>
</table>

Table 3.2 shows that the higher the magnitude of contribution of primary event $j$ to intensity $\lambda_t$ (i.e. the lower $\alpha$ value), the higher the mean. Table 3.3 indicates that the higher the primary event arrival rate (i.e. the higher $\rho$ value), the higher the mean. Table 3.4 shows that the lower the time period needed to determine events effects (i.e. the lower $\kappa$ value), the higher the mean.

**Example 3.3**
Using the same parameter values in Example 3.1, the calculations of the means of Cox with positive interest case at each value of the instantaneous rate $\delta$ are shown in Table 3.5.

<table>
<thead>
<tr>
<th>Table 3.5.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = 0.01$</td>
</tr>
<tr>
<td>7,960.1</td>
</tr>
<tr>
<td>$\delta = 0.03$</td>
</tr>
<tr>
<td>7,881.2</td>
</tr>
<tr>
<td>$\delta = 0.05$</td>
</tr>
<tr>
<td>7,803.3</td>
</tr>
<tr>
<td>$\delta = 0.07$</td>
</tr>
<tr>
<td>7,726.4</td>
</tr>
<tr>
<td>$\delta = 0.09$</td>
</tr>
<tr>
<td>7,650.9</td>
</tr>
</tbody>
</table>

**Example 3.4**
Let $t \rightarrow \infty$ in (3.5) and (3.7), then we have

$$
\frac{1}{\beta} \frac{\rho}{\alpha \kappa} \frac{1}{\delta}, \quad \text{(Shot noise Cox with positive interest with infinite time horizon)}
$$

and

$$
\frac{1}{\beta} \frac{\lambda}{\delta}, \quad \text{(Poisson with positive interest with infinite time horizon)}.
$$

Using the same parameter values in Example 3.1, Table 3.6 shows the actuarial net premiums of the discounted aggregate claims when the time horizon goes to $\infty$ for both shot noise Cox and Poisson case.
Table 3.6.

<table>
<thead>
<tr>
<th>Shot noise Cox with positive interest with infinite time horizon</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>160,000</td>
<td></td>
</tr>
</tbody>
</table>

4. Variations from the variance of double shot noise process

Similar to Section 3, if we set $\delta = 0$ in (2.24), it is given by

$$ m_2 \frac{\mu_1 \rho}{\kappa} \left( t + \mu_1 \frac{t}{\kappa} - \mu_1 \frac{1 - e^{-\kappa t}}{\kappa^2} \right) \quad (4.1) $$

as the time value of claims is no longer considered. The same result can also be found in Jang and Fu (2009). Replace $-\kappa$ with $\kappa$ in (1.6) and for simplicity, let us assume that $\lambda_0 = 0$, then (2.25) becomes

$$ \mu_2 \rho \left( \frac{e^{2\kappa t} - 1}{2\kappa} \right) = \mu_2 \rho \tilde{\gamma}_t \text{ at } 2\kappa. \quad (4.2) $$

Multiply $e^{-2\kappa t}$ in (4.2), then it becomes

$$ \mu_2 \rho \left( \frac{1 - e^{-2\kappa t}}{2\kappa} \right) = \mu_2 \rho \tilde{\gamma}_t \text{ at } 2\kappa. \quad (4.3) $$

Considering $\rho$ deterministic claim intensity, $\mu_1$ the mean of claim sizes and $\kappa$ the risk-free force of interest rate, then (4.2) and (4.3) are the variances of aggregate accumulated/discounted claims respectively where the claim arrival process follows a Poisson process. These results can also be found in Jang (2004).

Set $\kappa = 0$ either in (4.2) or in (4.3), then it is given by

$$ \mu_2 \rho t \quad (4.4) $$

as the time value of claims is no longer considered. It is the variance of aggregate claims where claim arrival process follows a Poisson process.

For the numerical comparison, let us denote $\rho = \lambda$, $\mu_1 = m_1$ and $\kappa = \delta$ in (4.2)-(4.4). Using an exponential distribution of $G(y)$ and $H(x)$ for event jump and claim size respectively, i.e.

$$ g(y) = \alpha e^{-\alpha y} \text{ and } h(x) = \beta e^{-\beta x} \text{ with } \alpha > 0, \beta > 0, $$

(2.24), (4.1), (4.3) and (4.4) are given by

$$ \frac{2}{\beta} \left[ \left\{ \frac{1}{\beta} \left( \frac{\rho^2}{\alpha^2 \kappa^2} + \frac{\rho}{\alpha^2 \kappa} \right) - \frac{1}{\beta \alpha^2 \kappa^2} \right\} \frac{1}{\delta - \kappa} - \frac{\rho^2}{\beta \alpha^2 \delta \kappa^2} \right] \left( \frac{1}{\delta + \kappa} \right) + \frac{2 \rho^2}{\beta^2 \alpha^2 \delta \kappa^2} \tilde{\gamma}_t \text{ at } \delta $$

$$ + \left[ \frac{2 \rho}{\beta^2 \alpha \kappa} - \frac{2}{\beta (\delta - \kappa)} \left\{ \frac{1}{\beta} \left( \frac{\rho^2}{\alpha^2 \kappa^2} + \frac{\rho}{\alpha^2 \kappa} \right) - \frac{1}{\beta \delta \alpha^2 \kappa} \right\} \tilde{\gamma}_t \text{ at } 2\delta $$

$$ - \frac{2}{\beta} \left[ \left\{ \frac{1}{\beta} \left( \frac{\rho^2}{\alpha^2 \kappa^2} + \frac{\rho}{\alpha^2 \kappa} \right) - \frac{1}{\beta \delta \alpha^2 \kappa} \right\} \frac{1}{\delta - \kappa} - \frac{\rho^2}{\beta \alpha^2 \delta \kappa^2} \right] \left( \frac{e^{-\delta (\delta + \kappa) t}}{\delta + \kappa} \right) $$

$$ - \left( \frac{\rho}{\beta \alpha \kappa} \tilde{\gamma}_t \text{ at } \delta \right)^2, \text{ (Shot noise Cox with positive interest)} \quad (4.5) \)
\[
\frac{2}{\beta^2} \frac{\rho}{\alpha \kappa} \left( t + \frac{t}{\alpha \kappa} - \frac{1 - e^{-\kappa t}}{\alpha \kappa^2} \right), \quad \text{(Shot noise Cox with no interest)} \tag{4.6}
\]

\[
\frac{2}{\beta^2} \lambda \bar{a}_{t|2\delta}, \quad \text{(Poisson with positive interest)} \tag{4.7}
\]

\[
\frac{2}{\beta^2} \lambda t. \quad \text{(Poisson with no interest)} \tag{4.8}
\]

In Section 3, we found that if \( \lambda = \frac{\rho}{\alpha \kappa} \), (3.5) and (3.7) (or (3.6) and (3.8)) have the same values, i.e.

\[
E^{\text{Poisson}}(L_t^0) = E^{\text{Cox}}(L_t^0).
\]

However, it does not hold in terms of the variances comparing between (4.5) and (4.7) (or between (4.6) and (4.8)). This implies that the distribution of the aggregate discount claims with respect to a Cox process has heavier tail than its counterpart with respect to a Poisson process, i.e.

\[
\text{Var}^{\text{Poisson}} \left\{ L_t^{(1)} \right\} < \text{Var}^{\text{Cox}} \left\{ L_t^{(1)} \right\}.
\]

Now let us illustrate the calculations of the variances of a compound Poisson model and their counterpart of a compound Cox model with/without considering a positive interest rate.

**Example 4.1**
The parameter values used to calculate the variances are

\[
\alpha = 0.5, \quad \rho = 4, \quad \kappa = 0.1, \quad \beta = 0.01, \quad \delta = 0.05, \quad t = 1
\]

and

\[
\lambda = \frac{\rho}{\alpha \kappa} = 80
\]

that provides us with the same means of aggregate discount claims regardless of the loss arrival process \( N_t \).

The calculations of the variances of a compound Poisson model and their counterpart of a compound Cox model with/without considering a positive interest rate are shown in Table 4.1.

Table 4.1.

<table>
<thead>
<tr>
<th></th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shot noise Cox with positive interest</td>
<td>2,995,400</td>
</tr>
<tr>
<td>Shot noise Cox with no interest</td>
<td>3,148,000</td>
</tr>
<tr>
<td>Poisson with positive interest</td>
<td>1,560,700</td>
</tr>
<tr>
<td>Poisson with no interest</td>
<td>1,600,000</td>
</tr>
</tbody>
</table>

Table 4.1 shows that even if the means of aggregate discount claims are the same regardless of the loss arrival process \( N_t \), the variance of aggregate discount claims from a compound Cox model is almost twice higher than its counterpart. Therefore if insurance companies employ mean-variance principle for their premium calculations, a compound Cox model offers higher premium than its counterpart. Table 4.1 justifies that insurance company can consider using a compound Cox model to accommodate stochastic nature of the claim arrivals from flood, windstorm, hail, bushfire and earthquake in practice.

**Example 4.2**
Using the same parameter values in Example 4.1, the calculations of variances of Cox with positive interest case caused by changes in the values of \( \alpha, \rho \) and \( \kappa \) are shown in Table 4.2-4.4.
Table 4.2 shows that the higher the magnitude of contribution of primary event \( j \) to intensity \( \lambda_t \) (i.e. the lower \( \alpha \) value), the higher the variance. Table 4.3 indicates that the higher the primary event arrival rate (i.e. the higher \( \rho \) value), the higher the variance. Table 4.4 shows that the lower the time period needed to determine events effects (i.e. the lower \( \kappa \) value), the higher the variance.

**Example 4.3**
Using the same parameter values in Example 4.1, the calculations of the variances of Cox with positive interest case at each value of the instantaneous rate \( \delta \) are shown in Table 4.5.

Example 4.4
Let \( t \rightarrow \infty \) in (4.5) and (4.7), then we have

\[
\begin{align*}
&\frac{2}{\beta} \left[ \frac{1}{\beta} \left( \frac{\rho^2}{\alpha^2 \kappa^2} + \frac{\rho}{\alpha^2 \kappa} \right) - \frac{1}{\beta \delta \alpha^2 \kappa} \right] \frac{1}{\delta - \kappa} - \frac{\rho^2}{\beta \alpha \kappa \delta} \right] \left( \frac{1}{\delta + \kappa} \right) + \frac{2\rho^2}{\beta^2 \alpha^2 \delta \kappa^2} \frac{1}{\delta} \\
&+ \frac{2\rho}{\beta^2 \alpha \kappa} - \frac{2}{\beta \left( \delta - \kappa \right)} \left[ \frac{1}{\beta} \left( \frac{\rho^2}{\alpha^2 \kappa^2} + \frac{\rho}{\alpha^2 \kappa} \right) - \frac{1}{\beta \delta \alpha^2 \kappa} \right] \left( \frac{1}{\beta \delta} \right) \\
&- \left( \frac{\rho}{\beta \alpha \kappa \delta} \right)^2,
\end{align*}
\]

(Shot noise Cox with positive interest with infinite time horizon) \( (4.9) \)

and

\[
\frac{1}{\beta^2} \lambda \frac{1}{\delta},
\]

(Poisson with positive interest with infinite time horizon). \( (4.10) \)

Using the same parameter values in Example 4.1, Table 4.6 shows the variances of the discounted aggregate claims when the time horizon goes to \( \infty \) for both shot noise Cox and Poisson case.

<table>
<thead>
<tr>
<th>Table 4.2</th>
<th>Table 4.3</th>
<th>Table 4.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 0.01 )</td>
<td>( \rho = 2 )</td>
<td>( \kappa = 0.01 )</td>
</tr>
<tr>
<td>( \alpha = 0.1 )</td>
<td>( \rho = 4 )</td>
<td>( \kappa = 0.1 )</td>
</tr>
<tr>
<td>( \alpha = 0.5 )</td>
<td>( \rho = 10 )</td>
<td>( \kappa = 0.5 )</td>
</tr>
<tr>
<td>( \alpha = 1 )</td>
<td>( \rho = 20 )</td>
<td>( \kappa = 1 )</td>
</tr>
<tr>
<td>( \alpha = 2 )</td>
<td>( \rho = 50 )</td>
<td>( \kappa = 2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \rho )</th>
<th>( \kappa )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3,758,100,000</td>
<td>1,497,700</td>
<td>30,398,000</td>
</tr>
<tr>
<td>44,433,000</td>
<td>2,995,400</td>
<td>2,995,400</td>
</tr>
<tr>
<td>2,995,400</td>
<td>7,488,500</td>
<td>564,000</td>
</tr>
<tr>
<td>1,129,500</td>
<td>14,977,000</td>
<td>264,270</td>
</tr>
<tr>
<td>472,700</td>
<td>37,442,000</td>
<td>119,340</td>
</tr>
</tbody>
</table>

Table 4.2

Table 4.3

Table 4.4

Table 4.5.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>3,116,700</td>
</tr>
<tr>
<td>0.03</td>
<td>3,055,300</td>
</tr>
<tr>
<td>0.05</td>
<td>2,995,400</td>
</tr>
<tr>
<td>0.07</td>
<td>2,937,000</td>
</tr>
<tr>
<td>0.09</td>
<td>2,880,000</td>
</tr>
</tbody>
</table>

Table 4.6.

<table>
<thead>
<tr>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shot noise Cox with positive interest with infinite time horizon</td>
</tr>
<tr>
<td>Poisson with positive interest with infinite time horizon</td>
</tr>
</tbody>
</table>
5 Conclusion

To accommodate stochastic nature of the claim arrivals from catastrophic events, such as flood, windstorm, hail, bushfire and earthquake, we considered a Cox process with shot noise intensity. For the additional economic assumption of a positive interest rate to claims, the duality result between the aggregate accumulated claims and single shot noise process was used. These two specifications made a double shot noise process and using its generator, we derived the moments of aggregate discounted claims. Removing the parameters in a double shot noise process, we showed that it became a compound Cox process with shot noise intensity, a single shot noise process and a compound Poisson process, respectively. Numerical comparisons were shown between the moments of a compound Poisson model and their counterparts of a compound Cox model with/without considering a positive interest rate. For that purpose, we assumed that claim sizes and primary event sizes follow an exponential distribution respectively.

References


