On the dynamic role of relative prices along the growth process*

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Abstract
We analyze the transitional dynamics of a model with heterogeneous consumption goods. In this model, convergence is driven by two different forces: the diminishing returns to capital and the growth of the relative price. We show that this second force affects the growth rate if the two consumption goods are not Edgeworth independent and if these two goods are produced with technologies that exhibit different capital intensities. Because this second force arises only when heterogeneous consumption goods are introduced, the transitional dynamics in this model exhibits interesting differences with respect to the transitional dynamics in a growth model with a unique consumption good. We show that these differences in the transitional dynamics can cause large discrepancies in the welfare cost of shocks between the economy with a unique consumption good and the economy with several consumption goods.

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1. Introduction

In this paper, we study the transitional dynamics of a multi-sector growth model where consumers’ utility depends on the consumption of heterogenous goods. As the main difference with the existing literature is the introduction of several consumption goods, we will focus our analysis of the transitional dynamics on the path followed by the rate of growth of consumption expenditure.

In growth models with a unique consumption good, the growth rate of consumption expenditures implied by the Euler equation only depends on the interest rate and convergence is thus determined only by the diminishing returns to capital (see Caballé and Santos, 1993; Lucas, 1988; Mulligan and Sala-i-Martín, 1993; Uzawa, 1965). In contrast, when several consumption goods are introduced, the growth rate of consumption expenditures depends on the interest rate and, in general, also on the growth of the relative prices. Therefore, convergence is governed by two different forces: the diminishing returns to capital and the growth of relative prices. Our aim is to show how this second force modifies the pattern of consumption growth.

The growth effect of the interest rate on consumption growth is measured by the intertemporal elasticity of substitution (IES) and the growth effect of an increase in the relative price between two consumption is measured by the Edgeworth elasticity of these two consumption goods. In fact, the effect on consumption growth of an increase in the relative price of the other consumption good depends on the sign of the Edgeworth elasticity. We show that when the two consumption goods are Edgeworth substitutes (complements) an increase in this price decreases (increases) the growth of consumption expenditure, and if the two consumption goods are Edgeworth independent then prices have no effect on consumption growth. The intuition is straightforward. The increase in the relative price of the other consumption good reduces the demand of the other consumption good, which increases (decreases) consumption growth when the two goods are substitutes (complements).

The literature has already analyzed the transitional dynamics of growth models with heterogeneous consumption goods. However, either they assume that the consumption goods are Edgeworth independent (see Echevarria, 1997; Laitner, 2000 or Perez and Guillo, 2010) or the assumptions on technology make the relative price between the two consumption goods constant (Kongsamunt, Rebelo and Xie, 2001; Ngai and Pissarides, 2007; and Steger, 2006). Therefore, to the best of our knowledge, this paper is the first one that studies the transitional dynamics of a growth model with heterogeneous consumption goods when the two forces driving the transition are operative.

In order to study the transitional dynamics when the second force is operative, we analyze a three sector growth model with an utility function that depends on a composite consumption good. This good is formed from combining two consumption goods that are produced using two capital stocks: physical and human capital. The different sectors produce using a constant returns to scale technology and the only difference between the technologies is the intensity of physical capital. However, this difference is crucial as it makes the relative price between the two consumption goods not constant along the transition. To gain some intuition about this result,}

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1Rebelo (2001) considers a model where the two forces are operative. However, he does not analyze the transitional dynamics.
suppose that human capital becomes relatively more scarce than physical capital. Then, the consumption good produced in the physical capital intensive sector becomes less costly and the relative price of this consumption good decreases. Note that if the consumption goods were produced with technologies with the same capital intensity then the imbalances between the two capital stocks would not modify the relative prices between these consumption goods and the second force would not be operative. We can then conclude that if the consumption goods are not Edgeworth independent and if they are produced with different technologies then the second force is operative so that the growth of prices modifies the growth rate of consumption expenditure along the transition.

As in any multi-sector growth model with two capital stocks, the transitional dynamics will be governed by the imbalances between these two capital stocks. However, the existence of two different forces driving the transition yields two interesting differences with respect to the transitional dynamics obtained in the standard growth model with a unique consumption good. First, in growth models with a unique consumption good, convergence in the consumption growth rate occurs from below (above) if the initial value of the ratio of physical to human capital is larger (smaller) than its stationary value. We show that this behavior may be reversed under heterogenous consumption goods. In particular, we provide a condition that implies that convergence is from above when the initial value of the ratio of capitals is larger than its stationary value and from below otherwise. Interestingly, when this condition is satisfied, the initial effect on consumption growth of a shock in one of the capital stocks will be the opposite to the one obtained in a model with a unique consumption good. As an example, consider an economy suffering a negative shock in human capital. Then, if there is a unique consumption good, this economy will suffer a reduction in consumption growth. In contrast, in our model with heterogeneous consumption, the economy will display an increase in consumption growth.

Second, while the growth rate of consumption exhibits a monotonic behavior in models where the only force governing the transition are the diminishing returns to capital, it may exhibit a non-monotonic behavior in our model. Alvarez-Cuadrado et al. (2004) mention evidence of non-monotonic behavior of the consumption growth rate. Steger (2000), among others, has explained this non-monotonic behavior with the introduction of a minimum consumption that makes preferences non-homothetic. In contrast, in our model this non-monotonic behavior is explained by the presence of the aforementioned two different forces driving the transitional dynamics. In fact, the growth rate exhibits a non-monotonic behavior when these two forces have opposite growth effects.

The two differences we have just mentioned imply that the patterns of growth along the transition strongly depend on the value of the parameters in our model with heterogeneous consumption goods. We simulate the economy in order to study the transition of the growth rate. We show that in the simulated economy the two forces have opposite growth effects. As a consequence, in the simulated economy the growth rate exhibits a non-monotonic convergence and the sign of the growth effects of a shock in one of the capital stocks depends on the value of the IES. We also use the simulated model to study the growth and welfare effects of technological and fiscal policy shocks. The aim of this analysis is to compare the effects of these shocks in
the economy with a unique consumption good with the effects in the economy with heterogeneous consumption goods. Regarding the welfare cost of shocks, we show that they will strongly depend on the sectoral composition of the composite consumption good when these shocks cause large effects on the cost of one unit of this composite consumption good. These large effects occurs when we consider shocks that modify the long run value of relative prices. We then conclude that existing literature, by considering specific models where the second force is not operative, obtain biased results on the effects of these shocks.

The paper is organized as follows. Section 2 outlines the model and characterizes the balanced growth path equilibrium. Section 3 studies the transitional dynamics and presents the main results. Section 4 develops the numerical analysis. Conclusions are summarized in Section 5 and the Appendix contains the proofs.

2. The economy

Let us consider a three-sector growth model in which the output in each sector is obtained from combining two types of capital, \( k \) and \( h \), which we denote physical and human capital, respectively. The first sector produces a commodity \( y \) using the following production function:

\[
y = A (s_y k)^\alpha (u_y h)^{1-\alpha} = Au_y h z_y^\alpha,
\]

where \( s_y \) and \( u_y \) are, respectively, the shares of physical and human capital allocated to this sector, \( z_y = s_y k / u_y h \) is the physical to human capital ratio, \( A > 0 \) is the sectoral total factor productivity (TFP), and \( \alpha \in (0,1) \) measures the intensity of physical capital in this sector. We assume that this sector produces manufactures and that the commodity \( y \) can be either consumed or added to the stock of physical capital. The law of motion of the physical capital stock is thus given by

\[
\dot{k} = Au_y h z_y^\alpha - c - \delta k, \tag{2.1}
\]

where \( c \) is the amount of good \( y \) devoted to consumption, and \( \delta \in [0,1] \) is the depreciation rate of the physical capital stock. The second sector produces consumption good \( x \) by means of the production function

\[
x = B (s_x k)^\beta (u_x h)^{1-\beta} = Bu_x h z_x^\beta, \tag{2.2}
\]

where \( s_x \) and \( u_x \) are the shares of physical and human capital allocated to this sector, respectively, \( z_x = s_x k / u_x h \) is the physical to human capital ratio, \( B > 0 \) is the sectoral TFP, and \( \beta \in (0,1) \) measures the intensity of physical capital in this sector. We assume that this sector produces food and services devoted to consumption, such as cultural or entertainment goods. Thus, the output of this sector can only be devoted to consumption. Finally, the third sector produces commodity \( e \) by means of the production function

\[
e = D [(1 - s_y - s_x) k]^{\sigma} [(1 - u_y - u_x) h]^{1-\sigma} = D (1 - u_y - u_x) h z_h^\pi,
\]

where \( z_h = (1 - s_y - s_x) k / (1 - u_y - u_x) h \) is the physical to human capital ratio, \( D > 0 \) is the sectoral TFP, and \( \pi \in (0,1) \) measures the intensity of physical capital
in this sector. This commodity is devoted exclusively to increase the stock of human capital and, therefore, we identify this sector with the education sector. The accumulation of the human capital stock is thus given by

\[ \dot{h} = D (1 - u_y - u_x) h z_h^\pi - \eta h, \]  

(2.3)

where \( \eta \in [0, 1] \) is the depreciation rate of human capital.

The economy is populated by an infinitely lived representative agent characterized by the following instantaneous utility function: \( U(c, x) \). We assume that the utility function is homothetic, concave and increasing in each argument. We also assume that population remains constant and that the representative agent is endowed with \( k \) units of physical capital and \( h \) units of human capital. Let \( w \) be the return from human capital (i.e., the real wage per unit of human capital) and \( r \) the return from physical capital (i.e., the real interest rate). We assume perfect sectoral mobility so that the wage and interest rate are independent of the sector where the representative agent allocates the units of physical and human capital. Accordingly, the consumer budget constraint is given by

\[ wh + rk = c + I_k + p_x x + p_h I_h, \]  

(2.4)

where \( p_x \) is the relative price of good \( x \) in units of good \( c \), \( p_h \) is the relative price of human capital in units of physical capital. Finally, \( I_h \) and \( I_k \) are the gross investment in human and physical capital, respectively,

\[ I_k = k + \delta k, \]  

(2.5)

and

\[ I_h = \dot{h} + \eta h. \]  

(2.6)

3. Equilibrium dynamics

In this section we first solve the consumers’ and firms’ problem and then we obtain the differential equations characterizing the equilibrium. We use these equations to obtain the long run equilibrium and to study how the introduction of a second consumption good modifies the time path of the growth rate of consumption expenditures.

The representative agent maximizes

\[ \int_0^\infty e^{-\rho t} U(c, x) dt, \]  

(3.1)

subject to (2.4), (2.5), and (2.6), where \( \rho > 0 \) is the subjective discount rate. The solution to this optimization problem is given by the following equations:²

\[ p_x = \frac{U_x}{U_c}, \]  

(3.2)

\[ \frac{\dot{p}_h}{p_h} = r - \frac{w}{p_h} + \eta - \delta, \]  

(3.3)

²The consumer’s maximization problem is solved in the appendix.
\[
\frac{\dot{c}}{c} = \left( \frac{\sigma_{cx} - \sigma_x}{\sigma_c \sigma_x - \sigma_{xc} \sigma_{cx}} \right) (r - \rho - \delta) - \left( \frac{\sigma_{cx}}{\sigma_c \sigma_x - \sigma_{xc} \sigma_{cx}} \right) \left( \frac{p_x}{p_c} \right),
\] (3.4)

and the transversality conditions

\[
\lim_{t \to \infty} e^{-\rho t} U_{ck} = 0,
\] (3.5)

and

\[
\lim_{t \to \infty} e^{-\rho t} U_{ch} = 0,
\] (3.6)

where \( U_i = \frac{\partial U}{\partial i}, \) \( U_{ii} = \frac{\partial^2 U}{\partial i^2}, \) \( \sigma_i = \frac{U_{ix}}{U_{ii}} \) and \( \sigma_{ij} = \frac{U_{ij}}{U_{ii}} \) for \( i = \{x, c\}, j = \{x, c\} \) and \( j \neq i \).

Equation (3.2) implies that the price level \( p_x \) is equal to the marginal rate of substitution between the two consumption goods. Equation (3.3) shows that the growth of the price \( p_h \) is determined by the standard non-arbitrage condition between investments in physical and human capital. Finally, (3.4) characterizes the growth rate of consumption good \( c \). Given that preferences are assumed to be homothetic, the growth rate of consumption good \( c \) coincides with the growth rate of consumption expenditure, where consumption expenditure is defined as \( \omega = c + p_x x \). For the sake of simplicity, our explanation on the analysis of the transition will be based on the growth rate of consumption good \( c \).

Equation (3.4) tells us that the growth rate of consumption expenditure is driven by both the interest rate and by the change in the relative price of the two consumption goods. The growth effect of a rise in the interest rate is measured by the intertemporal elasticity of substitution

\[
IES = \frac{\sigma_{cx} - \sigma_x}{\sigma_c \sigma_x - \sigma_{xc} \sigma_{cx}},
\]

and the growth effect of a rise in the growth rate of prices will depend on the elasticity of the marginal utility of the consumption good \( c \) with respect to the consumption good \( x \), \( \sigma_{cx} \). If \( \sigma_{cx} = 0 \) we claim that the two consumption goods are Edgeworth independent, if \( \sigma_{cx} < 0 \) we claim that they are Edgeworth substitutes and if \( \sigma_{cx} > 0 \) we claim that they are Edgeworth complements. Then, by using (3.4) and noting that concavity of the utility function implies that \( \sigma_c \sigma_x > \sigma_{xc} \sigma_{cx} \), we obtain the following result:

**Proposition 3.1.** If the two consumption goods are Edgeworth independent then the growth rate of prices does not directly affect the growth rate of consumption expenditure. In contrast, if the two consumption goods are Edgeworth substitutes (complements) then a rise in the relative price increases (decreases) consumption expenditure growth.

The intuition on this result is as follows. Equation (3.4) is the Euler equation that equals the market return from investing one unit of commodity \( y \) and the growth of the marginal utility obtained from consuming one additional unit of this commodity. When the two consumption goods are Edgeworth independent, then the marginal utility of consumption does not depend on the other consumption good. In this case, the growth rate of consumption only depends on the interest rate. In contrast, when the two
consumption goods are Edgeworth complementaries (substitutes) an increase in the consumption of good $x$ increases (decreases) the marginal utility of consumption good $c$. Thus, in this case, the growth of the marginal utility of consumption will depend on the growth of the two consumption goods. As follows from equation (3.2), the consumption of these goods depends on the relative price. Actually, concavity of the utility function implies that an increase in the relative price $p_x$ reduces the amount of good $x$ consumed. This reduction implies a reduction (increase) in the marginal utility of consumption good $c$ and in the amount of good $c$ consumed when the two goods are Edgeworth complementaries (substitutes). This mechanism then explains the result in Proposition 3.1.

After having presented the equilibrium conditions on the demand side of our economy, we will now move to the supply side and we characterize how the dynamics of relative prices are determined. This depends on the technologies used by the different sectors and on the market structure. In particular, firms maximize profits in each sector and, thus, the competitive factors payment must satisfy simultaneously the following conditions:

$$r = \alpha A z_y^{a-1},$$  \hspace{1cm} (3.7)
$$r = p_x \beta B z_x^{\beta-1},$$  \hspace{1cm} (3.8)
$$r = p_h \pi D z_h^{\pi-1},$$  \hspace{1cm} (3.9)
$$w = (1 - \alpha) A z_y^{a},$$  \hspace{1cm} (3.10)
$$w = p_x (1 - \beta) B z_x^{\beta},$$  \hspace{1cm} (3.11)

and

$$w = p_h (1 - \pi) D z_h^{\pi}. \hspace{1cm} (3.12)$$

Combining the system of equations (3.7) to (3.12), we obtain

$$z_i = \psi_i p_h^{\pi-1}, \hspace{0.5cm} i = y, h, x.$$  \hspace{1cm} (3.13)

where

$$\psi_y = \left(\frac{\pi}{\alpha}\right)^{\frac{\pi}{\alpha-1}} \left(\frac{1 - \alpha}{1 - \pi}\right)^{\frac{\pi}{\alpha-1}} \left(\frac{D}{A}\right)^{\frac{1}{\alpha-\pi}},$$

and

$$\psi_h = \left(\frac{\pi}{1 - \pi}\right) \left(\frac{1 - \alpha}{\alpha}\right) \psi_y,$$

and

$$\psi_x = \left(\frac{\beta}{1 - \beta}\right) \left(\frac{1 - \alpha}{\alpha}\right) \psi_y.$$  

Equation (3.13) shows that the three sector structure of our model nests two well-known growth models. On the one hand, when $\alpha = \beta$, the same technology produces the two consumption goods and the transitional dynamics of our model coincides with the transitional dynamics of the two-sector growth model with a unique consumption good, which was analyzed by Uzawa (1965) and Lucas (1988). In this case, (3.13) implies that $z_y = z_x$ and then, using (3.7) and (3.8), it follows that the relative price between the two consumption goods remains constant and equal to $p_x = A$. This means that
the growth of consumption only depends on the interest rate, as in the Uzawa-Lucas model.

On the other hand, when $\alpha = \pi$, the same technology produces the two capital stocks and the transition of our model coincides with the transition in models with several consumption goods and an AK technology. In this case, (3.13) implies that $z_y = z_h$. Then, using (3.7) and (3.9), it follows that the relative price between the two capital stocks is constant and equal to $p_h = \frac{1}{\beta}$. Equation (3.3), (3.7) and (3.10) imply that $z_y$ is constant when $p_h$ is constant. A constant value of $z_y$ implies that both the interest rate and $z_x$ are constant.\footnote{A constant interest rate implies that the technology is Ak. To see this, note that the technology that produces commodity $y$ can be rewritten as follows $y = \hat{A} u_y h$, where $\hat{A} = A (z_y^\alpha)$ is constant. The technology that produces commodity $x$ can be rewritten as $x = \hat{B} u_x h$, where $\hat{B} = B z_x^\beta$ is constant and, finally, the technology that produces commodity $e$ can be rewritten as $e = \hat{D} (1 - u_y - u_x) h$, where $\hat{D} = D (z_y^\alpha)$ is constant.} Equation (3.8) shows that in this case the relative price between the two consumption goods remains constant and the growth rate only depends on the interest rate.

We conclude from this analysis that the relative prices of consumption goods $p_x$ are constant when different sectors produce by using technologies with the same capital intensity. Therefore, in this case the growth rate of consumption expenditure are obviously only driven by the diminishing returns to capital as any growth model with a unique consumption good. Note that this conclusion does not depend on the particular assumption made on the number of capital stocks considered in the model. In fact, it would also arise in model with a unique capital stock. The heterogenous capital stock assumption will be used in what follows to determine the dynamics of prices. To this end, first we use equations (3.8), (3.9) and (3.13), to obtain

$$p_x = \varphi \frac{p_h}{p_x},$$

where

$$\varphi = \frac{\pi D (\psi_y)^{\pi - 1}}{\beta B (\psi_x)^{\beta - 1}}.$$  

This relationship between the relative prices implies that

$$\frac{\dot{p}_x}{p_x} = \left( \frac{\alpha - \beta}{\alpha - \pi} \right) \left( \frac{\dot{p}_h}{p_h} \right).  \tag{3.14}$$

Equation (3.14) shows that the relationship between the growth of the relative prices only depends on the capital intensity ranking among the different sectors.\footnote{Note that if $\beta = \pi$ then the consumption good $x$ and human capital are produced with the same technology and, in this case, the two relative prices satisfy $p_x = \frac{\hat{B}}{\hat{D}} p_h$.}

In order to simplify the characterization of the equilibrium, we consider the following functional form of the utility function:

$$U(c, x) = \frac{(c^{\theta} x^{1-\theta})^{1-\sigma}}{1-\sigma},$$

where the parameter $\theta \in [0, 1]$ measures the share of good $c$ in the composite consumption good $c^{\theta} x^{1-\theta}$, and $\sigma > 0$ is the (constant) elasticity of the marginal utility.
of this composite consumption good. Using this utility function, it is easy to show that \( IES = \frac{1}{\sigma}, \sigma_{cx} = (1 - \theta) (1 - \sigma) \) and \( \sigma_{c} \sigma_{x} - \sigma_{cx} \sigma_{cx} = \sigma > 0 \).

We next characterize the shares of physical and human capital in each sector. To this end, we define the aggregate ratios \( z = k/h \) and \( q = c/k \). Then, we combine (2.2) and (3.2) to get

\[
\begin{align*}
    u_x &= \left(1 - \frac{\theta}{\theta}\right) \left(\frac{q z}{p_x B z_{x}^{\beta}}\right) , \quad (3.15)
    \\
    s_x &= \left(1 - \frac{\theta}{\theta}\right) \left(\frac{q z_{x}^{1-\beta}}{p_x B}\right) . \quad (3.16)
\end{align*}
\]

and we use the definition of \( z_x \) to obtain

\[
\begin{align*}
    s_x &= \left(1 - \frac{\theta}{\theta}\right) \left(\frac{z x}{p_{x} B z_{x}^{\beta}}\right) , \quad (3.16)
\end{align*}
\]

Next, we combine the definitions of \( z_y \) and \( z_h \) to get

\[
\begin{align*}
    u_y &= \left(1 - u_x\right) z_h - (1 - s_x) z_y , \quad (3.17)
    \\
    s_y &= \left(\frac{z_y}{z}\right) \left(\frac{1 - u_x}{z_h - z_y}\right) . \quad (3.18)
\end{align*}
\]

We proceed to characterize the growth rate of the two capital stocks. For that purpose, we use (2.1) to obtain

\[
\begin{align*}
    \frac{\dot{k}}{k} &= \frac{A u_y z_{y}^\alpha}{z} - q - \delta , \quad (3.19)
    \\
    \frac{\dot{h}}{h} &= D (1 - u_y - u_x) z_{h}^\pi - \eta . \quad (3.20)
\end{align*}
\]

Finally, we obtain the equations that characterize the equilibrium path. First, we combine (3.3), (3.7) and (3.10) to obtain

\[
\begin{align*}
    \frac{\dot{p}_h}{p_h} &= \frac{\alpha A z_{y}^{\alpha - 1} - (1 - \alpha) A z_{y}^{\alpha}}{\kappa(p_h)} + \eta - \delta . \quad (3.21)
\end{align*}
\]

Note that the right hand side of the previous equation can be written as a function of the relative price \( p_h, \kappa(p_h) \), after making use of (3.13).

We combine (3.4) with (3.7) and (3.14) to obtain

\[
\begin{align*}
    \frac{\dot{c}}{c} &= \frac{\alpha A z_{y}^{\alpha - 1} - \rho - \delta}{\sigma} - \chi(p_h) \equiv \gamma(p_h) , \quad (3.22)
    \\
    \text{where} \quad \chi &= \left(\frac{(1 - \sigma)(1 - \theta)}{\sigma}\right) \left(\frac{\alpha - \beta}{\alpha - \pi}\right) .
\end{align*}
\]
Note that the first term in the right hand side of (3.22) is a function of the relative price \( p_h \), \( \nu (p_h) \), as follows from (3.13). Equation (3.22) shows the two forces governing the transition and the parameters measuring the intensity of these two forces. In particular, the growth effects of an increase in prices are measured by the parameter \( \chi \). This parameter can take either positive or negative values depending on the two consumption goods being substitutes or complements and depending on the relative factor intensity ranking among the three sectors. Thus, an increase in the growth rate of the relative prices can increase or decrease the consumption growth rate depending on the sign of \( \chi \).

Combining (3.19) and (3.20), we get
\[
\frac{\dot{z}}{z} = \frac{Au_y \gamma^\alpha}{z} - \eta - \delta - D (1 - u_y - u_x) z^\pi_h,
\]
and combining (3.19) and (3.22) we obtain
\[
\frac{\dot{q}}{q} = \nu (p_h) - \chi \kappa (p_h) - \frac{Au_y \gamma^\alpha}{z} + \eta + \delta.
\]

The dynamic equilibrium is thus characterized by a set of paths \( \{p_h, z, q\} \) such that, given the initial value of the ratio between the two capital stocks \( z_0 \), solves the equations (3.21), (3.23), and (3.24), and satisfies (3.13), (3.15), (3.16), (3.17) together with the transversality conditions (3.5) and (3.6). As in the standard two-sector growth model, there is a unique state variable, \( z \), and the transition will be governed by the imbalances between the two capital stocks.

**Remark 1.** Let us combine (3.21), (3.22), (3.7) and (3.10) to obtain that the rate of growth of consumption satisfies
\[
\frac{\dot{c}}{c} = \gamma (p_h) = \left( \frac{1}{\sigma} - \chi \right) r + \left( \frac{\chi}{p_h} \right) w - \frac{\rho + \delta}{\sigma} - \chi (\eta - \delta). \tag{3.25}
\]
This equation shows that the rate of growth of consumption depends both on the interest rate and on the wage rate when \( \chi \neq 0 \). From this equation we obtain two interesting implications for empirical analysis. First, cross-country differences in the growth rates will also be explained by wage differentials when \( \chi \neq 0 \). Moreover, for values of \( \chi \) close to the IES, interest rate differentials will not explain cross country differences in the growth rates. Second, wages are closely related to current income. Thus, our version of the infinite horizon model implies that the consumption growth rate will depend on current income. Obviously, the permanent income hypothesis holds in our model. We conclude then that obtaining a positive empirical relationship between current income and consumption growth does not necessary imply the failure of the permanent income hypothesis.

We define a steady-state equilibrium as an equilibrium path along which the ratios \( z \) and \( q \) and the relative prices of goods and capitals remain constant. The following result characterizes the steady-state equilibrium.
Proposition 3.2. The unique long-run value $p_h^*$ of the relative price solves $\kappa (p_h^*) = 0$, the two capital stocks and consumption expenditure grow at the same constant growth rate $g^* \equiv \nu (p_h^*)$, and the long run value $z^*$ of the ratio of capitals and the long run value $q^*$ of the ratio of consumption to capital are unique.

Note that neither the long-run price level $p_h^*$ nor the growth rate $g^*$ depend on the parameter $\theta$ measuring relative weight of the consumption goods in the utility function. As in the standard endogenous growth model with a unique consumption good, the long-run values of these two variables only depend on the technology. In contrast, the long run value of the ratios $z^*$ and $q^*$ depend on the parameter $\theta$.\(^5\) On the one hand, as $\theta$ increases, the weight of consumption good $c$ in the utility function increases and, as a consequence, the ratio $q^*$ increases in the long run. On the other hand, the change in the patterns of consumption due to an increase in $\theta$ also affects the long run value of the ratio of capitals $z^*$. In particular, when the sector that produces the consumption good $c$ is relatively more (less) intensive in physical than the sector that produces the consumption good $x$, a rise in $\theta$ increases (decreases) the demand of physical capital in comparison to the demand of human capital and then the ratio $z^*$ increases (decreases) with $\theta$.\(^6\)

Let us now analyze how the behavior of the growth rate of consumption expenditure during the transition is affected by the introduction of a second consumption good.

Proposition 3.3. The steady state equilibrium is locally saddle path stable.

This result implies that the dynamic equilibrium is unique, which allows us to make comparisons between the growth patterns and to define the concept of asymptotic speed of convergence. Concerning the asymptotic speed of convergence, in the proof of Proposition 3.3 it is shown that if $\alpha > \pi$ then the asymptotic speed of convergence is equal to $p_h^* \kappa'(p_h^*)$ and is independent of the parameter $\theta$. In contrast, if $\alpha < \pi$ then the asymptotic speed of convergence depends on this parameter. In this case, the equilibrium value of the relative price $p_h$ equals its steady state value and is then constant along the transition. This implies that the growth rate of consumption expenditure is constant and equal to $v(p_h)$ along the transition when $\alpha < \pi$. Therefore, there is no convergence in terms of expenditure growth in this case. Following Perli and Sakellaris (1998), we will assume that $\alpha > \pi$ so that expenditure growth will exhibit transitional dynamics.\(^7\)

We proceed with the analysis of the two different forces governing the transition in this economy. It is important to note that this dynamic analysis is global in the sense that the conclusions obtained from this analysis hold even when the equilibrium path is far from the steady state. As shown in equation (3.22), these two forces are summarized in the terms $v(p_h)$ and $\kappa(p_h)$, which are functions of the relative price. The function $v(p_h)$ summarizes the growth effects of an increase in the interest rate and $\kappa(p_h)$ is a measure of the growth effects of the relative price. These two functions

\(^5\)The exact expressions of $z^*$ and $q^*$ are displayed in the appendix.

\(^6\)The proof of these results is available upon request.

\(^7\)The role of the factor intensity ranking in the transitional dynamics of multi-sector growth models is extensively discussed in Bond et al. (1996).
are decreasing when $\alpha > \pi$.\footnote{In the proof of Proposition 3.2 we show that $\kappa (p_h)$ is decreasing when $\alpha > \pi$ and, using (3.13), it is immediate to see that $\nu (p_h)$ is also a decreasing function in this case.} As the two forces only depend on the relative price, the nature of the transition will depend on the slope of the stable manifold relating the price $p_h$ with the state variable $z$, as it determines the dynamic adjustment of relative prices along the transition. We proceed to characterize this dynamic adjustment.

**Lemma 3.4.** The stable manifold relating prices and the ratio of capitals is monotonic.

The intuition behind this lemma is as follows. When $z_0 < (>) z^\ast$, $h_0$ is large (small) in comparison to $k_0$ and then the relative price of human capital will be lower (higher) than its long run value. This implies that the relative price $p_h$ decreases along the transition when $z_0 > z^\ast$ and increases otherwise. Obviously, this means that the slope of the stable manifold is positive and it also means that $\kappa (p_h) < (>) 0$ when $z_0 > (>) z^\ast$.

**Proposition 3.5.** The ratio of capitals, $z$, exhibits a globally monotonic transition.

The result in Proposition 3.5 allows us to characterize analytically the global transitional dynamics of the growth rate of consumption expenditure. We should first mention that the coexistence of two forces determining the transition implies that the dynamic path of this variable may exhibit non-monotonocities when these two forces have opposite growth effects. To show these non-monotonocities, we use (3.25), (3.7) and (3.10) to obtain the following derivative of the rate of growth of consumption expenditure with respect to the relative price $p_h$:

$$
\frac{\partial \gamma (p_h)}{\partial p_h} = \left( 1 - \frac{\alpha}{\pi} \right) \phi (p_h),
$$

(3.26)

where

$$
\phi (p_h) = \pi \chi \psi y p_h^{1 - \frac{n + \sigma}{n - \pi}} - \alpha \left( \frac{1}{\sigma} - \chi \right).
$$

Note that if $\chi \in (0, 1/\sigma)$ then there exists a value of $p_h$ such that $\phi (p_h) = 0$. As the price monotonically increases with $z$, there is a value of $z$, say $\bar{z}$, such that $\phi (p_h) > (>) 0$ when $z > (>) \bar{z}$. The following result uses these arguments and Proposition 3.5 to provide conditions for the existence of non-monotonic behavior and to characterize the global transition dynamics of the growth rate of consumption expenditure:

**Proposition 3.6.** Assume that $\theta \in (0, 1)$. Then,

(a) If $\chi \leq 0$, the time path of the growth rate of consumption expenditures is monotonically decreasing (increasing) when $z_0 < z^\ast$ ($z_0 > z^\ast$).

(b) If $\chi \in (0, 1/\sigma)$ and $\bar{z} < z^\ast$, the time path of the growth rate of consumption expenditures monotonically decreases when $z_0 > z^\ast$, monotonically increases when $z_0 \in (\bar{z}, z^\ast)$, and it exhibits a non-monotonic behavior when $z_0 < \bar{z}$.

(c) If $\chi \in (0, 1/\sigma)$ and $\bar{z} \geq z^\ast$, the time path of the growth rate of consumption expenditures monotonically decreases when $z_0 < z^\ast$, monotonically increases when $z_0 \in (z^\ast, \bar{z})$, and it exhibits a non-monotonic behavior when $z_0 > \bar{z}$.
If $\chi > 1/\sigma$, the time path of the growth rate of consumption expenditures is monotonically increasing (decreasing) when $z_0 < z^*$ ($z_0 > z^*$).

The results in Proposition 3.6 imply that in this economy we can distinguish four types of transition. These different types of transition are represented in Figure 1, where the growth rate of consumption expenditure is displayed as a function of the ratio of capitals. In particular, Panel i shows the growth rate of consumption expenditure when $\chi = 0$ and growth of consumption expenditure is not affected by the growth of the relative price $p_h$. In this case, as in the Uzawa-Lucas model, the growth rate of consumption expenditure is a monotonic function that decreases when $z_0 < z^*$ and increases when $z_0 > z^*$ (see Mulligan and Sala-i-Martín (1993) for a complete analysis of the transitional dynamics of the Uzawa-Lucas model). In fact, $\chi = 0$ when the production structure of the economy coincides with the one in the Uzawa-Lucas model ($\alpha = \beta$), there is a unique consumption good ($\theta = 1$), or the two consumption goods are Edgeworth independent ($\sigma = 1$). Moreover, the same type of convergence holds when $\chi < 0$. However, when $\chi \in (0, 1/\sigma)$ the two forces governing the transition have opposite growth effects and the patterns of growth are different from the ones in the Uzawa-Lucas model. On the one hand, the growth rate of consumption expenditure exhibits a non-monotonic behavior when the initial value of the ratio of capitals is sufficiently far from its stationary value. On the other hand, as shown in Panels ii and iii, we must distinguish two types of transition, depending on the relationship between $z$ and $z^*$. Interestingly, when $\zeta < z^*$, the local dynamics implies that convergence is from below when $z_0 < z^*$ and from above otherwise. Therefore, in this case, the conclusions from convergence are reversed due to the effect of the growth of prices. As shown in Panel iv, this reversed transition also arises when $\chi > 1/\sigma$. To see the implications of this reversed transition, suppose that the economy suffers a reduction in the stock of physical capital that reduces the ratio $z$ of physical to human capital. This reduction implies an initial increase in the growth rate of consumption expenditure in a model with a unique consumption good, whereas it implies an initial reduction in the growth rate in our model.

4. Numerical Analysis

The results in Proposition 3.6 imply that the transition crucially depends on the value of the parameters. We next discuss which is the most plausible type of transition. We address this issue through the following simulation. In order to fit our model with data, we will consider that the commodity $y$ is manufacturing, the consumption good $x$ is composed of primary goods and services, and human capital is education. We use the labor income shares in the primary, manufacturing and service sectors, and the sectoral composition of GDP reported by Echevarria (1997) for the US economy to set $\alpha = 0.34$ and $\beta = 0.49$. We take the average share of physical capital in the final education output estimated by Perli and Sakellaris (1998) and we consider $\pi = 0.18$.

---

9The value of $\beta$ is a weighted average of the capital income shares in the agriculture (0.71%) and service (0.49%) sectors in the US and the weights are the fraction of GDP in agriculture (1.7%) and in services (72.2%). These weights are obtained from NIPA.
We assume $\delta = 0.056$ to replicate that the investment in physical capital amounts to 7.6% of its stock. Moreover, Perli and Sakellaris (1998) pointed out that estimates of the depreciation rate $\eta$ vary widely. We choose $\eta = 0.025$, which corresponds with the low end of the range. We set arbitrarily $A = B = 1$, and set $D = 0.0851$ to generate a long-run interest rate net of depreciation equal to 5.6%. The parameter $\theta$ measures the fraction of total consumption expenditures devoted to consumption goods produced in the manufacturing sector. According to Kongsamunt et al. (2001), this fraction was roughly constant during the last century and equal to 0.3. We then select this value for the parameter $\theta$. The value of the other two preference parameters, $\sigma$ and $\rho$, depends on the value of the IES. As the Proposition 3.6 shows, this value crucially determines the nature of the transition, as it provides a measure of the intensity of the first force. We consider three different values of $IES: 0.5$, $0.41$ and $0.37$. We set the values of $\sigma$ and $\rho$ that jointly replications these values for $IES$ and a long-run growth rate equal to 2%. In the high elasticity economy we obtain $\sigma = 2$ and $\rho = 0.016$, whereas we get $\sigma = 2.4$ and $\rho = 0.008$ for the $IES = 0.41$ economy, and we get $\sigma = 2.7$ and $\rho = 0.002$ for the low elasticity economy.

4.1. Transitional dynamics

The expression of the parameter $\chi$ implies that it takes positive values if $\alpha < \beta$ and $IES < 1$. Thus, $\chi$ is positive when we choose empirically plausible values of the parameters. In this case, the two forces governing the transition have opposite growth effects. In our numerical examples, we show that if the second force dominates then the transition is going to be different from the one in models with a unique consumption good. Figures 2, 3 and 4 show that this occurs when the IES is low.

[Insert Figures 2, 3 and 4]

Figures 2, 3 and 4 compare the transitional dynamics of the economy with heterogeneous consumption goods (continuous line) with the transition in an equivalent economy with a unique consumption good, that is $\theta = 1$ (dashed line). Each figure contains six panels. Panels i, iv and v display, respectively, the growth rate of consumption expenditure, the growth rate of GDP and the speed of convergence of the state variable, $z$, as a function of the deviations of the capital ratio with respect to its stationary value.\textsuperscript{10} Panels ii and iii display, respectively, the time path of the growth rate of consumption expenditure when the state variable is initially below its long run value and when it is initially above. Finally, the continuous line in Panel vi shows the welfare cost of a reduction in the stock of physical capital as a function of the parameter $\theta$ and the dotted line in the same panel shows the welfare gain of an increase in this capital stock.\textsuperscript{11}

Figure 2 shows the transitional dynamics in the high elasticity economy. The growth rate of consumption expenditure as a function of the ratio of capitals exhibits an U-shaped curve when $\theta = 0.3$, which means that the transition is non-monotonic

\textsuperscript{10}We follow Reiss (2000) and we define the non asymptotic speed of convergence of the ratio of capitals as $\frac{\dot{z}}{z}$.

\textsuperscript{11}As in Lucas (1987), we measure the welfare cost (gain) of a shock by the percentage increase (decrease) in consumption necessary to keep welfare unaffected by the shock.
We see that the non-monotonic behavior arises in the economy with heterogeneous consumption when the initial value of the ratio of capital is above its long run value. Panels ii and iii compare the time paths in this economy with the time paths in an equivalent economy with a unique consumption good, that is $\theta = 1$. From this comparison, it follows that in both economies convergence is from above when the ratio of capitals is initially smaller than its long run value and it is from below otherwise.

Figure 3 shows the transitional dynamics in the $IES = 0.41$ economy. The transition of the growth rate of consumption expenditures is also non-monotonic. However, in this case, when the ratio of capitals is initially smaller than its long run value, the growth rate of consumption expenditures converges in our economy with two consumption goods from below, whereas converges from above in the economy with a unique consumption good. When the ratio of capital is initially above its long run value, the growth rate of consumption expenditures converges from below in our economy with heterogeneous consumption and from above in the economy with $\theta = 1$. Therefore, when the IES is low, the introduction of heterogeneous consumption goods reverses the transition. This occurs because the $IES$ measures the growth effects of the interest rate. Then, when the $IES$ is sufficiently low, the growth effect of changes in the interest rate is low in comparison with the growth effects of changes in the growth of the relative price. In this case, even if the initial values of the economy are close to the corresponding steady-state values, the transition is different from the one arising in an economy where the transition is governed only by the diminishing returns to capital. This reversed transition of the growth rate of consumption expenditures is also displayed in Figure 4 that shows the transitional dynamics in the $IES = 0.37$ economy. In this case, the IES is so low that the second force dominates the transition. This implies that the transition is monotonic but reversed when heterogeneous consumption goods are introduced (see Panels i, ii and iii).

As follows from Figures 2, 3 and 4 the path of the GDP growth rate, the speed of convergence and the welfare cost is nearly independent of the value of $IES$. This happens because in the numerical exercises changes in $IES$ imply changes in $\rho$ aimed to keep the value of the long run growth rate unaffected. In contrast, these figures show that the path of these variables depends on the value of the parameter $\theta$. This parameter measures the weight of the human capital intensive consumption good in the composite consumption good. Thus, a reduction in $\theta$ makes the composite consumption good more physical capital intensive, which explains the results displayed in these three panels. Panel iv shows that if physical capital is scarce in comparison to human capital, so that $z < z^*$, then the growth rate of GDP is larger when $\theta = 0.3$. This result is a consequence of the fact that the composite consumption good is more physical capital intensive when $\theta = 0.3$. This implies a faster accumulation of physical capital when $\theta = 0.3$, which explains the larger growth of GDP. The same intuition explains that if $z > z^*$ then the growth of GDP is smaller when $\theta = 0.3$. In this case, physical capital is relatively abundant and its stock relative to human capital is reduced along the transition. This reduction will be faster when the composite consumption good is more physical capital intensive. This faster reduction of physical capital explains that the economy with $\theta = 0.3$ exhibits lower growth rates of GDP. Moreover, the fact that the transition in the stock of physical capital relative to human capital is faster when the composite consumption good is more physical capital intensive implies that the
non-asymptotic speed of convergence decreases with $\theta$ (See panel v). Finally, Panel vi shows that a reduction in physical capital that places the economy out of the BGP, in the region where $z < z^*$, will cause a larger welfare cost in the economy with a lower $\theta$. The same panel shows that an increase in the capital stock that places the economy in the region where $z > z^*$ will cause a larger welfare gain in the economy with a lower $\theta$. Again, these results follow from the fact that the composite consumption good in the economies with a low value of $\theta$ is more physical capital intensive. Obviously, in these economies welfare depends more on the value of the stock of capital.

Table 1 shows the uniform increase (decrease) in consumption necessary to compensate the welfare loss (gains) obtained by individuals from a reduction (increase) in the physical capital stock both in an economy with a unique consumption good and in an economy with two consumption goods. The last column of this table compares the differences in welfare costs between these two economies and shows that they are large. In particular, the welfare cost is approximately a 25% larger in the economy with two consumption goods, whereas the welfare gain is 8% larger in this economy. Interestingly, different numerical exercises show that these differences, as a percentage, are robust to shocks of different size.

4.2. Comparative dynamics and welfare

We proceed to study the transitional dynamics and the welfare costs of four different shocks: biased technological shock, unbiased technological shock, the introduction of an income tax and the introduction of a capital income tax. The aim of this analysis is to compare the effects of these shocks in an economy with heterogeneous consumption goods with the effects in an economy with a unique consumption good.

We first study the growth effects of a biased technological shock that consists of a 25% permanent reduction in the TFP of the manufacturing sector, given by the parameter $A$. Figure 5 displays the consumption growth rate in our economy with two consumption goods (continuous line) and the growth rate of consumption expenditures in an economy with a unique consumption good (dashed line). The first panel compares these growth rates when the $IES = 0.5$. In both economies the patterns of growth implied by these technological shocks are similar. The growth rate initially suffers a strong decline and then it increases until it converges to its new long run value. Obviously, this long run value is smaller than the one before the shock. In the economy with a unique consumption good, the growth rate only depends on the interest rate, which falls due to the technological shock. This reduces investment and then the stock of physical capital declines during the transition. The reduction in the stock of physical capital implies that the interest rate increases during the transition. Note that the behavior of the interest rate explains the initial strong reduction in the growth rate and also the increase in the growth rate during the transition. In the economy with heterogeneous consumption goods, the growth rate also depends on the growth of the relative price between the two capital stocks. As the stock of physical capital decreases during the transition, the relative price of human capital falls. This causes an increase in the growth rate as $\chi > 0$. This positive growth effect explains that the reduction in the growth rate, shown in the first panel of Figure 5, is smaller in the economy with heterogeneous consumption goods than in the economy with a unique consumption good.
good. This positive growth effect also explains the growth effects of the shock when $IES = 0.41$ and $IES = 0.37$. In these two cases, in the economy with heterogeneous consumption goods the growth rate initially increases and then decreases until the long run growth rate is attained. This initial positive growth effect is explained by the growth effects of the prices that initially dominate the transition when the IES is sufficiently low.

[Insert Figures 5, 6 and 7]

Table 2 shows the uniform increase in consumption necessary to compensate the welfare cost of a permanent reduction in the TFP of the manufacturing sector. The last column of this table compares the welfare cost of this shock in an economy with a unique consumption good and in an economy with two consumption goods. As follows from the table, the welfare cost of this shock is twice in the economy with a unique consumption good. Obviously, this large difference is a consequence of the biased shock that only reduces the TFP of the consumption good produced in the manufacturing sector. The reduction in TFP implies an increase in the relative price of the consumption good produced in the manufacturing sector. As this is the only good consumed when $\theta = 1$, this shock causes a larger increase in the cost of one unit of the composite consumption good in this economy than in the economy with heterogeneous consumption goods. This explains the large differences in the welfare costs reported in Table 2.

Figure 5 also displays the growth effects of an unbiased technological shock that consists of a permanent 10% reduction of the TFP in each sector. The growth effects of this shock are similar to the ones due to the biased technological shock when $\theta = 1$. However, as each sectoral TFP is reduced by the same amount, the effect on relative prices is smaller than the effect due to the biased technological shock. Obviously, this implies that the second force is smaller when shocks are unbiased. This explains that the path of the growth rate in the economy with two consumption goods is similar to the path in the economy with a unique consumption good. This small effect of prices also implies that the discrepancies in the welfare cost of this shock between the two economies are small (see Table 3). In fact, using (3.13) and (3.21), it can be shown that if we had assumed equal depreciation rates of the two capital stocks then this shock would not change the long run value of the relative price between the two consumption goods. This would imply that the real cost of the composite good is not permanently affected by this shock. In this case, the discrepancies in the welfare cost between the two economies would be almost zero. Therefore, the small differences in the welfare cost of the shock reported in Table 3 are just a consequence of the gap between the depreciation rates of the two capital stocks. In contrast, the biased technological shocks reduces the long run value of the relative price between the two consumption goods, even if we assume equal depreciation rates. Thus, this shock reduces the real cost of the composite consumption good in the economy with $\theta = 0.3$, while it does not affect the long run real cost of the consumption good in the economy with $\theta = 1$. The different effects of this shock on the real cost of the composite consumption good explain the large differences in the welfare cost shown in Table 2. We then conclude that large discrepancies in the welfare cost of shocks between the two economies exists when the shocks modify the value of the relative prices between the different consumption goods.
Figure 6 displays the growth effects of the introduction of a 10% income tax rate. We assume that tax revenues are returned to households through a lump-sum subsidy in order to prevent wealth effects. The introduction of this tax reduces the returns from capital, which explains the initial large reduction in the growth rate. Two forces determine the transition towards the new steady state equilibria. On the one hand, the introduction of this tax reduces the accumulation of capital, which causes an increase in the interest rate and in the growth rate during the transition. On the other hand, the return on both types of capital decreases, which drives the change in prices. When $\theta = 1$, this second force is not operative and then the growth rate increases during the transition. However, when $\theta = 0.3$, the two forces are operative and have opposite growth effects. As a consequence, the two forces compensate and the growth rate is roughly constant along the transition.

Table 4 shows the welfare cost of the introduction of this income tax. The introduction of this tax causes an intertemporal substitution effect that implies that during the transition there is a welfare gain associated to a temporal increase in consumption. However, there is a welfare cost in the BGP. Obviously, the welfare cost in the BGP is larger than the temporal welfare gain and thus the introduction of this tax causes a welfare cost once the entire equilibrium path is considered. From the comparison between the economy with a unique consumption good and the economy with two consumption goods, it follows that there are no relevant differences in the welfare cost once the entire path is considered. This is a consequence of the fact that the introduction of income taxes only has minor effects on the value of the relative price between the two consumption goods. In fact, using (3.13) and (3.21), it can be shown that there would be no long run effects if we had assumed the same depreciation rate of the two capital stocks. Figure 6 also displays the welfare cost of the introduction of this tax as a function of $\theta$ and it shows that the sectoral composition of the composite consumption good has almost no effect on the welfare cost of this shock.

Figure 7 displays the growth effects of the introduction of a 10% capital income tax rate when tax revenues are returned to households through a lump-sum subsidy. This tax has similar effects on the interest rate than the income tax rate. However, this tax reduces the return on physical capital, while it does not modify the return on human capital. As a consequence, the relative prices between the two capital stocks and between the two consumption goods will be directly modified by the introduction of this tax. This larger effect on the relative price between the two capital stocks implies that the second force driving convergence is more intensive and, for a sufficiently low $IES$, initially dominates the transition. This explains the transition of the growth rate of consumption expenditures displayed in Figure 7. The effect on the relative price between the two consumption goods implies that the welfare cost associated to the introduction of this tax will depend on the sectoral composition of the composite consumption good. More precisely, this capital income tax increases the relative price of the consumption good produced in the capital intensive sector. As a consequence, the welfare cost of the introduction of this tax will be larger in those economies where the composite consumption good is more capital intensive. In our numerical example, the composite consumption good is more physical capital intensive when the parameter $\theta$ takes a lower value. This conclusion is illustrated in Table 5 that compares the welfare cost of the introduction of a 10% capital income tax and in Figure 7 that displays this
welfare cost as a function of $\theta$.

5. Concluding remarks

We have analyzed the transitional dynamics of an endogenous growth model with two consumption goods. We have shown that if the two consumption goods are not Edgeworth independent and the technologies producing the two consumption goods have different capital intensities then consumption growth not only depends on the interest rate, but also on the growth of the relative price between the two consumption goods. In this case, convergence is determined by two different forces: the diminishing returns to capital and the growth of prices. The existence of this second force yields interesting differences with respect to the transitional dynamics obtained in the standard growth model with a unique consumption good. We illustrate these differences using a growth model with two capital stocks that we identify with human and physical capital. First, we show that in contrast with the standard growth model, convergence in the growth rate may occur from above if the initial value of the ratio of physical to human capital is larger than its stationary value and may occur from below otherwise. Second, we show that the growth rate of consumption expenditures may exhibit a non-monotonic behavior when the two forces have opposite growth effects. Moreover, these differences in the transition have interesting implications.

On the one hand, economies with the same interest rate may exhibit different growth rates of consumption along the transition. Therefore, our model provides an additional explanation to cross-country differences in the growth rates. Rebelo (1992) shows that the introduction of a minimum consumption requirement also implies that the growth rates do not equalize. This occurs because the minimum consumption makes preferences be non-homothetic and then the IES is not constant along the transition. In this framework, convergence is driven by the interest rate and by the time-varying IES. More recently, Steger (2006) shows that, if there are heterogenous consumption goods and a unique capital stock, then the IES is not constant and the growth rates do not equalize. Obviously, he derives this result when preferences are non-homothetic. In contrast, we show that, when there are heterogenous consumption goods, the growth rates are different even with a constant IES because of the effect of the growth of the relative capital prices along the transition.

On the other hand, the welfare cost of shocks will depend on the sectoral composition of the composite consumption good. The relationship between the welfare cost of shocks and the sectoral composition of the composite consumption good will be particularly strong when the shocks modify the value of relative prices. In this case, the effect of these shocks on the cost of the composite consumption good will depend on its sectoral composition. We have shown that biased technological shocks and tax policies that increase the gap between the return on physical and human capital cause large and permanent effects on prices. We have also shown that the welfare cost of these shocks depends on the sectoral composition of the composite consumption good.

We conclude that the results obtained in aggregate growth models with a unique consumption good cannot be generalized to more disaggregated models with heterogenous consumption goods. In these disaggregated models, the welfare costs of shocks depends on the value of the parameters measuring the sectoral composition of
the composite consumption good and on the physical capital intensities of the sectors producing these consumption goods. Therefore, future research on the welfare cost of shocks should be concerned with the value of these parameters.

An interesting extension of this paper is to introduce a minimum consumption requirement in one of the consumption goods. The price of this good will be high in the initial stages of development since the minimum consumption requirement will induce a high marginal utility of this good. Then, as the economy develops, the price will fall sharply until convergence is attained. Therefore, it seems that the introduction of a minimum consumption may accelerate the change of prices and, hence, the introduction of this consumption requirement may increase the effect of the growth of the relative price on both the growth rate of consumption expenditures and on the welfare cost of shocks.
References


A. Appendix

Solution to the consumer’s optimization problem.

The Hamiltonian function associated to the maximization of (3.1) subject to (2.4), (2.5) and (2.6) is

\[ H = e^{-\rho t} u(c, x) + \lambda (wh + r k - c - I_k - p_x x - p_h I_h) + \mu_1 (I_k - \delta k) + \mu_2 (I_h - \eta h), \]

where \( \lambda, \mu_1, \) and \( \mu_2 \) are the co-state variables corresponding to the constraints (2.4), (2.5) and (2.6), respectively. The first order conditions are

\[ e^{-\rho t} u_c - \lambda = 0, \quad \text{(A.1)} \]
\[ e^{-\rho t} u_x - \lambda p_x = 0, \quad \text{(A.2)} \]
\[ \lambda = \mu_1, \quad \text{(A.3)} \]
\[ p_h \lambda = \mu_2, \quad \text{(A.4)} \]
\[ \lambda r - \delta \mu_1 = -\dot{\mu}_1, \quad \text{(A.5)} \]
\[ \lambda w - \eta \mu_2 = -\dot{\mu}_2. \quad \text{(A.6)} \]

Combining (A.1) and (A.2), we obtain (3.2) and

\[ \frac{\dot{x}}{x} = \left( \frac{\sigma_{cx} - \sigma_x}{\sigma_{cx} - \sigma_x} \right) \frac{\dot{c}}{c} - \left( \frac{1}{\sigma_{cx} - \sigma_x} \right) \frac{\dot{p}_x}{p_x}, \quad \text{(A.7)} \]

where \( \sigma_i = \frac{U_i \mu}{U_i} \), and \( \sigma_{ij} = \frac{U_i}{U_j} \) for \( i = x, c, \) and \( j = x, c \) and \( j \neq i \). Using (A.3) and (A.4), we obtain that

\[ p_h \mu_1 = \mu_2, \]

which implies that

\[ \frac{\dot{p}_h}{p_h} + \frac{\dot{\mu}_1}{\mu_1} = \frac{\dot{\mu}_2}{\mu_2}, \]

and (3.3) follows from using (A.5) and (A.6). Combining (A.1), (A.3) and (A.5), we obtain that

\[ -r + \delta = -\rho + \sigma_c \left( \frac{\dot{c}}{c} \right) + \sigma_{cx} \left( \frac{\dot{x}}{x} \right), \]

and (3.4) follows from using (A.7). Finally, the transversality conditions (3.5) and (3.6) follow from combining (A.1), (A.3) and (3.2).

Proof of Proposition 3.2. The uniqueness of \( p_h^\ast \) follows from the monotonicity of \( \kappa(p_h) \), which can be shown using (3.21)

\[ \kappa'(p_h) = - \left( \alpha + \frac{\pi z_y}{p_h} \right) \left( \frac{(1 - \alpha) Az_y^{\alpha-1}}{(\alpha - \pi) p_h} \right) > (\ast) \quad \text{if} \quad \alpha < (\ast) \pi. \]
Combining (3.15), (3.16) and (3.17), we obtain

\[
    u_y = \frac{z_h - z}{z_h - z_y} + \left(1 - \frac{\theta p_x B z_x^2}{\varepsilon}\right) \left(\frac{z_x - z_h}{z_h - z_y}\right) q z
\]  

(A.8)

and

\[
    1 - u_y - u_x = \frac{z - z_y}{z_h - z_y} + \varepsilon \left(\frac{z_y - z_x}{z_h - z_y}\right) q z.
\]  

(A.9)

Next, in a steady state, equations (3.20) and (3.19) simplify to

\[
    1 - u_y - u_x = \frac{g^* + \eta}{P z_h^\alpha},
\]

\[
    Au_y z^\alpha - q = g^* + \delta.
\]

These two equations can be rewritten as the following system of two equations by using (A.8) and (A.9):

\[
    z + \varepsilon (z_y - z_x) q z = \left(\frac{g^* + \eta}{P z_h^\alpha}\right) (z_h - z_y) + z_y,
\]

\[
    z_h + \left(\varepsilon (z_x - z_h) - \frac{z_h - z_y}{A z_y^\alpha}\right) z q = \left(\frac{z_h - z_y}{A z_y^\alpha}\right) \left(\frac{g^* + \delta}{A z_y^\alpha}\right) + 1 z.
\]

The steady state values of \(z^*\) and \(q^*\) are the unique solution of this system of equations and they are equal to

\[
    z^* = \frac{\varepsilon \tilde{z} (z_x - z_h) + \varepsilon (z_y - z_x) z_h - \varepsilon \left(\frac{z_h - z_y}{A z_y^\alpha}\right)}{\varepsilon (z_x - z_h) + \tilde{z} \varepsilon (z_y - z_x) - \frac{z_h - z_y}{A z_y^\alpha}},
\]

and

\[
    q^* = \frac{\tilde{z} \varepsilon - z_h}{\varepsilon \tilde{z} (z_x - z_h) - \tilde{z} \left(\frac{z_h - z_y}{A z_y^\alpha}\right) + \varepsilon (z_y - z_x) z_h},
\]

where

\[
    \varepsilon = \frac{(1 - \theta)}{\theta p_x B z_x^2},
\]

\[
    \tilde{z} = \left(\frac{g^* + \eta}{D z_h^\pi}\right) (z_h - z_y) + z_y,
\]

and

\[
    \tilde{\varepsilon} = (z_h - z_y) \left(\frac{g^* + \delta}{A z_y^\alpha}\right) + 1.
\]
Proof of Proposition 3.3. Let \( J \) be the Jacobian matrix evaluated at the steady state of the system of differential equations formed by (3.21), (3.23) and (3.24),

\[
J = \begin{pmatrix}
\frac{\partial \dot{p}_h}{\partial p_h} & \frac{\partial \dot{p}_h}{\partial z} & \frac{\partial \dot{p}_h}{\partial q} \\
\frac{\partial \dot{z}}{\partial p_h} & \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial q} \\
\frac{\partial \dot{q}}{\partial p_h} & \frac{\partial \dot{q}}{\partial z} & \frac{\partial \dot{q}}{\partial q}
\end{pmatrix},
\]

where

\[
\frac{\partial \dot{p}_h}{\partial p_h} = p_h \kappa' (p_h),
\]

\[
\frac{\partial \dot{p}_h}{\partial z} = 0,
\]

\[
\frac{\partial \dot{p}_h}{\partial q} = 0,
\]

\[
\frac{\partial \dot{z}}{\partial p_h} = z \left\{ \frac{\partial^2 u_y}{\partial p_h^2} \left( \frac{\partial u_y}{\partial z} \right) + \frac{\partial u_y \alpha z_y^{\alpha - 1}}{\partial z_h} \left( \frac{\partial z_y}{\partial p_h} \right) \right\},
\]

\[
\frac{\partial \dot{z}}{\partial z} = \left\{ -Dz_h^\pi \left( \frac{\partial (1 - u_y - u_x)}{\partial p_h} \right) - D (1 - u_y - u_x) \pi z_h^{\pi - 1} \left( \frac{\partial z_h}{\partial p_h} \right) \right\},
\]

\[
\frac{\partial \dot{z}}{\partial q} = \left\{ -Az_y^\sigma \left( \frac{\partial u_y}{\partial z} \right) - 1 - Dz_h^\pi \left( \frac{\partial (1 - u_y - u_x)}{\partial q} \right) \right\},
\]

\[
\frac{\partial \dot{q}}{\partial p_h} = q \left\{ \frac{\alpha (\alpha - 1)}{\sigma} \left( \frac{\partial z_y}{\partial p_h} \right) - \chi \kappa' (p_h) - \epsilon_q \right\},
\]

and

\[
\frac{\partial \dot{q}}{\partial z} = -q \epsilon_z,
\]

\[
\frac{\partial \dot{q}}{\partial q} = -q \epsilon_q.
\]

The determinant of the Jacobian matrix is

\[
Det(J) = \frac{\partial \dot{p}_h}{\partial p_h} \left( \frac{\partial \dot{z}}{\partial z} \frac{\partial \dot{q}}{\partial q} - \frac{\partial \dot{z}}{\partial q} \frac{\partial \dot{q}}{\partial z} \right) = q \kappa' (p_h) p_h Dz_h^\pi M,
\]

25
where

\[
M = \epsilon_y \left( \frac{\partial (1 - u_y - u_x)}{\partial z} \right) - \epsilon_z \left( \frac{\partial (1 - u_y - u_x)}{\partial q} \right) =
\]

\[
= \left[ \frac{\partial u_y}{\partial z} - \frac{A u_y z^\alpha}{z^2} \left( \frac{\partial u_y}{\partial q} \right) \right] - \left[ \frac{A z^\alpha_y}{z} \left( \frac{\partial u_y}{\partial q} \right) - 1 \right] \frac{\partial u_x}{\partial z} + \left[ - \frac{A u_y z^\alpha_y}{z^2} + \frac{A z^\alpha_y}{z} \left( \frac{\partial u_y}{\partial q} \right) \right] \frac{\partial u_x}{\partial q}.
\]

Using (3.17) and (3.18), \( M \) simplifies to

\[
M = -\left( \frac{1}{z_h - z_y} \right) \left[ 1 + A z^\alpha_y - \varepsilon z_x + \varepsilon (z_y - z_x) \left( A z^\alpha_y - g^* - \delta \right) \right],
\]

and the determinant satisfies

\[
\text{Det} (J) = -\left( \frac{z g \kappa'(p_h) p_h D z^\pi_h}{z_h - z_y} \right) \left[ 1 + A z^\alpha_y - \varepsilon (z_y - z_x) \left( g \frac{\sigma - \alpha}{\alpha} + \rho + \delta (1 - \alpha) \right) \right] < 0,
\]

as \( \kappa'(p_h) > (\kappa) < 0 \) when \( z_h > (\kappa) < z_y \). Next, we obtain the value of the trace

\[
\text{Tr} (J) = \frac{\partial \dot{p}_h}{\partial p_h} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{q}}{\partial q} =
\]

\[
= p_h \kappa'(p_h) + \left[ A z^\alpha_y \left( \frac{\partial u_y}{\partial z} \right) - \frac{A u_y z^\alpha_y}{z^2} - D \frac{\partial (1 - u_y - u_x)}{\partial z} z_h^\pi \right] - q \left[ A z^\alpha_y \left( \frac{\partial u_y}{\partial q} \right) - 1 \right].
\]

Using (3.17) and (3.18), the trace simplifies to

\[
\text{Tr} (J) = \alpha A z^\alpha_y - \frac{(1 - \alpha) A z^\alpha_y}{p_h} - (g + \eta) - (g + \delta).
\]

Using \( \kappa = 0 \), we obtain

\[
\text{Tr} (J) = 2 [(\sigma - 1) g + \rho] > 0,
\]

as follows from the transversality condition.

Therefore, the trace is positive, whereas the determinant is negative. This means that there is a unique negative root and that the equilibrium is saddle path stable. Note that the negative root is \( p_h \kappa'(p_h) \) when \( \alpha > \pi \). Otherwise, the negative roots is one of the roots obtained from the reduced system of differential equations formed by equations (3.23) and (3.24).

Proof of Lemma 3.4. Equation (3.13) shows that the physical to human capital ratio in the three sectors, \( z_y \), \( z_x \) and \( z_h \), depends positively on the relative price \( p_h \) when \( \alpha > \pi \). Hence, there is a positive relation between the capital ratio \( z \) and the relative prices \( p_h \) along the transition dynamics.

Proof of Proposition 3.5 In the proof of Proposition 3.3, we show that \( \kappa'(p_h) \) does not change its sign. This means that relative prices exhibit a monotonic transition. Lemma 3.4 states that the stable manifold relating prices and the ratio of capitals is monotonic. This implies that the ratio of capitals must exhibit a monotonic behavior along the entire transition.
Proof of Proposition 3.6. Parts (a) and (d) follow since the relative price \( p_h \) is a monotonic and increasing function of \( z \) and \( \phi(p_h) > 0 \) when \( \chi \leq 0 \) and \( \phi(p_h) < 0 \) when \( \chi > \frac{1}{\beta} \). In Part (b), as the slope of the stable manifold is positive, \( \phi(p_h) > 0 \) along the transition when \( z_0 > z^* \) and changes its sign when \( z_0 < \tau < z^* \). In the first case, the consumption growth rate is monotonically decreasing, whereas it exhibits a non-monotonic behavior when \( z_0 < \tau \). In particular, if \( z_0 < \tau \) the growth rate initially decreases and then, as the economy converges to the steady state, it increases. In Part (c), \( \phi(p_h) > 0 \) along the transition when \( z_0 < z^* \) and changes its sign when \( z_0 > \tau \geq z^* \). In the first case, the consumption growth rate is monotonically decreasing, whereas it exhibits a non-monotonic behavior when \( z_0 > \tau \). In particular, if \( z_0 > \tau \) the growth rate initially decreases and, as the economy converges to the steady state, it increases. \( \blacksquare \)
Figure 1. Growth rate of expenditure
Figure 2. $IES = 0.5$
Figure 3. IES = 0.41
Figure 4. \( IES = 0.37 \)
Figure 5. Growth effects of a technological shock
Figure 6. Growth effects of an income tax
Figure 7. Growth effects of a capital income tax
Table 1. Welfare cost

<table>
<thead>
<tr>
<th>$I E S$</th>
<th>$\theta = 0.3$ (A)</th>
<th>$\theta = 1$ (B)</th>
<th>$A/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>38.00%</td>
<td>30.34%</td>
<td>1.25</td>
</tr>
<tr>
<td>0.41</td>
<td>38.09%</td>
<td>30.39%</td>
<td>1.25</td>
</tr>
<tr>
<td>0.37</td>
<td>38.14%</td>
<td>30.39%</td>
<td>1.26</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$I E S$</th>
<th>$\theta = 0.3$ (A)</th>
<th>$\theta = 1$ (B)</th>
<th>$A/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$-63.27%$</td>
<td>$-58.70%$</td>
<td>1.08</td>
</tr>
<tr>
<td>0.41</td>
<td>$-63.22%$</td>
<td>$-58.68%$</td>
<td>1.08</td>
</tr>
<tr>
<td>0.37</td>
<td>$-63.18%$</td>
<td>$-58.67%$</td>
<td>1.08</td>
</tr>
</tbody>
</table>

Table 2. Welfare cost of a biased technological change

<table>
<thead>
<tr>
<th>$I E S$</th>
<th>$\theta = 0.3$ (A)</th>
<th>$\theta = 1$ (B)</th>
<th>$A/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>26.47%</td>
<td>51.06%</td>
<td>0.52</td>
</tr>
<tr>
<td>0.41</td>
<td>26.61%</td>
<td>51.21%</td>
<td>0.52</td>
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<tr>
<td>0.37</td>
<td>26.68%</td>
<td>51.30%</td>
<td>0.52</td>
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Table 3. Welfare cost of an unbiased technological change

<table>
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<tr>
<th>$I E S$</th>
<th>$\theta = 0.3$ (A)</th>
<th>$\theta = 1$ (B)</th>
<th>$A/B$</th>
</tr>
</thead>
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<tr>
<td>0.5</td>
<td>30.11%</td>
<td>29.61%</td>
<td>1.02</td>
</tr>
<tr>
<td>0.41</td>
<td>30.48%</td>
<td>29.98%</td>
<td>1.02</td>
</tr>
<tr>
<td>0.37</td>
<td>30.70%</td>
<td>30.20%</td>
<td>1.02</td>
</tr>
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</table>
Table 4. Welfare cost of an income tax

<table>
<thead>
<tr>
<th>$\tau = 0.1$</th>
<th>$\theta = 0.3$ (A)</th>
<th>$\theta = 1$ (B)</th>
<th>$A/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IES$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Entire path</td>
<td>1.48%</td>
<td>1.48%</td>
<td>1.00</td>
</tr>
<tr>
<td>0.5 Adjustment path</td>
<td>-6.78%</td>
<td>-7.03%</td>
<td>0.96</td>
</tr>
<tr>
<td>BGP path</td>
<td>11.74%</td>
<td>11.04%</td>
<td>1.06</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tau_k = 0.1$</th>
<th>$\theta = 0.3$ (A)</th>
<th>$\theta = 1$ (B)</th>
<th>$A/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IES$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Entire path</td>
<td>1.27%</td>
<td>1.27%</td>
<td>1.00</td>
</tr>
<tr>
<td>0.41 Adjustment path</td>
<td>-6.01%</td>
<td>-5.83%</td>
<td>1.03</td>
</tr>
<tr>
<td>BGP path</td>
<td>8.99%</td>
<td>9.60%</td>
<td>0.94</td>
</tr>
</tbody>
</table>

Table 5. Welfare cost of a capital income tax

<table>
<thead>
<tr>
<th>$\tau_k = 0.1$</th>
<th>$\theta = 0.3$ (A)</th>
<th>$\theta = 1$ (B)</th>
<th>$A/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IES$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Entire path</td>
<td>0.46%</td>
<td>0.42%</td>
<td>1.08</td>
</tr>
<tr>
<td>0.5 Adjustment path</td>
<td>-1.33%</td>
<td>-1.32%</td>
<td>1.08</td>
</tr>
<tr>
<td>BGP path</td>
<td>3.06%</td>
<td>2.95%</td>
<td>1.03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tau_k = 0.1$</th>
<th>$\theta = 0.3$ (A)</th>
<th>$\theta = 1$ (B)</th>
<th>$A/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IES$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Entire path</td>
<td>0.44%</td>
<td>0.42%</td>
<td>1.08</td>
</tr>
<tr>
<td>0.41 Adjustment path</td>
<td>-1.05%</td>
<td>-1.04%</td>
<td>1.00</td>
</tr>
<tr>
<td>BGP path</td>
<td>2.62%</td>
<td>2.59%</td>
<td>1.03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tau_k = 0.1$</th>
<th>$\theta = 0.3$ (A)</th>
<th>$\theta = 1$ (B)</th>
<th>$A/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IES$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Entire path</td>
<td>0.14%</td>
<td>0.41%</td>
<td>1.07</td>
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<tr>
<td>0.37 Adjustment path</td>
<td>-0.89%</td>
<td>-0.89%</td>
<td>1.00</td>
</tr>
<tr>
<td>BGP path</td>
<td>2.37%</td>
<td>2.28%</td>
<td>1.03</td>
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</tbody>
</table>