Chapter 2: Response distributions

2.1 *Using the notation of Section 2.2, show that the mean and variance of a Bernoulli random variable are π and $\pi(1-\pi)$ respectively. More generally show that the mean and variance of binomial random variable are $n\pi$ and $n\pi(1-\pi)$, respectively.*

Bernoulli:

$$
\begin{aligned}
f(y) &= \pi^y(1-\pi)^{1-y} \qquad y = 0, 1 \\
\mathrm{E}(y) &= \sum_{j=0}^{1} j \cdot \pi^j (1-\pi)^{1-j} \\
&= 0 \cdot \pi^0(1-\pi)^1 + 1 \cdot \pi^1(1-\pi)^0 \\
&= \pi \ . \\
\mathrm{E}(y^2) &= \sum_{j=0}^{1} j^2 \cdot \pi^j (1-\pi)^{1-j} \\
&= \pi \ , \text{ and hence} \\
\mathrm{Var}(y) &= \mathrm{E}(y^2) - \{\mathrm{E}(y)\}^2 \\
&= \pi - \pi^2 \\
&= \pi(1-\pi) \ .
\end{aligned}
$$

Binomial:

$$
\begin{aligned}
f(y) &= \binom{n}{y}\pi^y(1-\pi)^{n-y} \qquad y=0,\ldots,n \\
\mathrm{E}(y) &= \sum_{j=0}^{n} j\cdot\binom{n}{j}\pi^j(1-\pi)^{n-j} \\
&= \sum_{j=1}^{n} j\frac{n!}{j!(n-j)!}\pi^j(1-\pi)^{n-j} \\
&= n\pi\sum_{j=1}^{n}\frac{(n-1)!}{(j-1)!(n-j)!}\pi^{j-1}(1-\pi)^{n-j} \\
&= n\pi\sum_{k=0}^{n-1}\binom{n-1}{k}\pi^k(1-\pi)^{n-1-k} \qquad \text{putting } k=j-1 \\
&= n\pi\{\pi+(1-\pi)\}^{n-1} \qquad \text{using the binomial theorem} \\
&= n\pi\ . \\
E(y^2) &= \sum_{j=0}^{n} j^2\cdot\binom{n}{j}\pi^j(1-\pi)^{n-j} \\
&= n\pi\sum_{j=1}^{n} j\cdot\frac{(n-1)!}{(j-1)!(n-j)!}\pi^{j-1}(1-\pi)^{n-j} \\
&= n\pi\sum_{k=0}^{n-1}(k+1)\binom{n-1}{k}\pi^k(1-\pi)^{n-1-k} \\
&= n\pi\left\{\sum_{k=0}^{n-1}k\binom{n-1}{k}\pi^k(1-\pi)^{n-1-k}+\sum_{k=0}^{n-1}\binom{n-1}{k}\pi^k(1-\pi)^{n-1-k}\right\} \\
&= n\pi\{(n-1)\pi)+1\}\ . \\
\mathrm{Var}(y) &= \mathrm{E}(y^2)-\{\mathrm{E}(y)\}^2 \\
&= n\pi\{(n-1)\pi)+1\}-(n\pi)^2 \\
&= n\pi(1-\pi)\ .
\end{aligned}
$$

2.2 *Show that the mean and variance of a χ^2_ν random variable are ν and 2ν respectively.*

The pdf of the χ^2_ν distribution is

$$
f(y)=\frac{1}{2^{\nu/2}\Gamma(\frac{\nu}{2})}y^{\frac{\nu}{2}-1}\mathrm{e}^{-\frac{y}{2}}\ .
$$

$$
\begin{aligned}
\mathrm{E}(y) &= \frac{1}{2^{\nu/2}\Gamma(\frac{\nu}{2})}\int_0^\infty u\cdot u^{\frac{\nu}{2}-1}\mathrm{e}^{-\frac{u}{2}}\,du \\
&= \frac{1}{2^{\nu/2}\Gamma(\frac{\nu}{2})}\int_0^\infty u^{\frac{\nu}{2}}\mathrm{e}^{-\frac{u}{2}}\,du \\
&= \frac{1}{2^{\nu/2}\Gamma(\frac{\nu}{2})}\int_0^\infty (2z)^{\frac{\nu}{2}}\mathrm{e}^{-z}\,2\,dz \qquad \text{putting } z=\frac{u}{2} \\
&= \frac{2^{\nu/2+1}}{2^{\nu/2}\Gamma(\frac{\nu}{2})}\int_0^\infty z^{\frac{\nu}{2}}\mathrm{e}^{-z}\,dz \\
&= \frac{2\Gamma(\frac{\nu}{2}+1)}{\Gamma(\frac{\nu}{2})} \\
&= \frac{2\cdot\frac{\nu}{2}\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu}{2})} \\
&= \nu\ .
\end{aligned}
$$

Using similar logic, we can show that $\mathrm{E}(y^2)=\nu(\nu+2)$ and $\mathrm{Var}(y)=\nu(\nu+2)-\nu^2=2\nu$.

2.3 *The distribution of the number of failures y till the first success in independent Bernoulli trials, with probability of success π at each trial, is the geometric:*

$$f(y)=(1-\pi)^y\pi \qquad y=0,1,\ldots\ .$$

Show that the mean and variance of the geometric distribution are $\mathrm{E}(y)=(1-\pi)/\pi$ *and* $\mathrm{Var}(y)=(1-\pi)/\pi^2$.

$$
\begin{aligned}
\mathrm{E}(y) &= \sum_{j=0}^\infty j\cdot(1-\pi)^j\pi \\
&= \pi\cdot\frac{1-\pi}{\pi^2} \qquad \left(\text{using } \sum_{k=0}^\infty kr^k=\frac{r}{(1-r)^2}\right) \\
&= \frac{1-\pi}{\pi}\ . \\
\mathrm{E}(y^2) &= \sum_{j=0}^\infty j^2\cdot(1-\pi)^j\pi \\
&= \pi\cdot\frac{(1-\pi)(2-\pi)}{\pi^3} \qquad \left(\text{using } \sum_{k=0}^\infty k^2r^k=\frac{r(1+r)}{(1-r)^3}\right) \\
&= \frac{(1-\pi)(2-\pi)}{\pi^2}\text{ , giving} \\
\mathrm{Var}(y) &= \frac{(1-\pi)(2-\pi)}{\pi^2}-\left(\frac{1-\pi}{\pi}\right)^2 \\
&= \frac{1-\pi}{\pi^2}\ .
\end{aligned}
$$