Chapter 5: Generalized Linear Models

5.1 *Show that* $\mathrm{E}\{\dot\ell(\beta)\} = 0$.

$$\dot\ell(\beta) = \frac{\partial \ln f(y)}{\partial\beta} = \frac{1}{f(y)}\frac{\partial f(y)}{\partial\beta} \qquad \text{using the chain rule.} \tag{1}$$

Its expected value is

$$\begin{aligned}
\mathrm{E}\{\dot\ell(\beta)\} &= \int \dot\ell(\beta) f(y)\,dy \\
&= \int \frac{1}{f(y)}\frac{\partial f(y)}{\partial\beta} f(y)\,dy \\
&= \int \frac{\partial f(y)}{\partial\beta}\,dy \\
&= \frac{\partial}{\partial\beta}\int f(y)dy \\
&= 0 \qquad \text{since} \int f(y)dy = 1\ .
\end{aligned}$$

5.2 *Show that* $\mathcal{I} = \mathrm{E}\{\dot\ell(\beta)\dot\ell'(\beta)\}$.

We assume firstly that β is a scalar. In Exercise 5.1, we showed that $\mathrm{E}\{\dot\ell(\beta)\} = 0$. Therefore $\frac{\partial}{\partial\beta}\mathrm{E}\{\dot\ell(\beta)\} = 0$, and

$$\begin{aligned}
0 &= \frac{\partial}{\partial\beta}\int \dot\ell(\beta) f(y)\,dy \\
&= \int \frac{\partial}{\partial\beta}\Big\{\dot\ell(\beta)f(y)\Big\}\,dy \qquad \text{(exchanging the order of integration and differentiation)} \\
&= \int \left\{\ddot\ell(\beta)f(y) + \dot\ell(\beta)\frac{\partial f(y)}{\partial\beta}\right\} dy \\
&= \int \ddot\ell(\beta)f(y)dy + \int \Big\{\dot\ell(\beta)\Big\}^2 f(y)dy \qquad \left(\text{substituting } \frac{\partial f(y)}{\partial\beta} = \dot\ell(\beta)f(y) \text{ from (1)}\right) \\
&= \mathrm{E}\{\ddot\ell(\beta)\} + \mathrm{E}[\{\dot\ell(\beta)\}^2]\ .
\end{aligned} \tag{2}$$

When β has dimension p, then $\dot\ell(\beta)$ is of length p, and equation (2) becomes

$$0 = \mathrm{E}\{\ddot\ell(\beta)\} + \mathrm{E}\{\dot\ell(\beta)\dot\ell(\beta)'\}$$

and

$$\mathcal{I} = -\mathrm{E}\{\ddot\ell(\beta)\} = \mathrm{E}\{\dot\ell(\beta)\dot\ell(\beta)'\}\ .$$

5.3 *Show that the deviances of the binomial, gamma, inverse Gaussian and negative binomial distributions are as given in Table 5.2.*

We will need the following (from Section 3.4):

	θ	$a(\theta)$	$\dot a(\theta)$	$\check\theta_i$	$\hat\theta_i$	ϕ
Binomial	$\ln\frac{\pi}{1-\pi}$	$n\ln(1+\mathrm{e}^\theta) = -n\ln(1-\theta)$	$\frac{n\mathrm{e}^\theta}{1+\mathrm{e}^\theta}$	$\ln\frac{y_i}{n_i-y_i}$	$\ln\frac{\hat\pi_i}{1-\hat\pi_i}$	1
Gamma	$-1/\mu$	$-\ln(-\theta) = \ln\mu$	$-1/\theta$	$-1/y_i$	$-1/\hat\mu_i$	$1/\nu$
Inverse Gaussian	$-\frac{1}{2\mu^2}$	$-\sqrt{-2\theta} = -\frac{1}{\mu}$	$\frac{1}{\sqrt{-2\theta}}$	$-\frac{1}{2y_i^2}$	$-\frac{1}{2\hat\mu_i^2}$	σ^2
Negative binomial	$\ln\frac{\kappa\mu}{1+\kappa\mu}$	$-\frac{1}{\kappa}\ln(1-\mathrm{e}^\theta) = \frac{1}{\kappa}\ln(1+\kappa\mu)$	$\frac{1}{\kappa}\frac{\mathrm{e}^\theta}{1-\mathrm{e}^\theta}$	$\ln\frac{\kappa y_i}{1+\kappa y_i}$	$\ln\frac{\kappa\hat\mu_i}{1+\kappa\hat\mu_i}$	1

Equation (5.11) states

$$\Delta = 2\sum_{i=1}^{n}\left\{\frac{y_i(\check\theta_i - \hat\theta_i) - a(\check\theta_i) + a(\hat\theta_i)}{\phi}\right\} .$$

Binomial

$$\begin{aligned}
\Delta &= 2\sum_{i=1}^{n}\frac{y_i(\ln\frac{y_i}{n_i-y_i} - \ln\frac{\hat\pi_i}{1-\hat\pi_i}) - n_i\ln(1+\frac{y_i}{n_i-y_i}) + n_i\ln(1+\frac{\hat\pi_i}{1-\hat\pi_i})}{\frac{1}{n_i}} \\
&= 2\sum_{i=1}^{n} n_i\left\{y_i\ln\frac{y_i}{n_i-y_i} - y_i\ln\frac{\hat\pi_i}{1-\hat\pi_i} + n_i\ln(n_i-y_i) - n_i\ln(1-\hat\pi_i)\right\} \\
&= 2\sum_{i=1}^{n} n_i\left\{y_i\ln\frac{y_i}{\hat\pi_i} + (n_i-y_i)\ln\frac{n_i-y_i}{1-\pi_i}\right\} .
\end{aligned}$$

Gamma

$$\begin{aligned}
\Delta &= 2\sum_{i=1}^{n}\frac{y_i(-\frac{1}{y_i}+\frac{1}{\hat\mu_i}) - \ln y_i + \ln\hat\mu_i}{\frac{1}{\nu}} \\
&= 2\nu\sum_{i=1}^{n}\left\{\frac{y_i-\hat\mu_i}{\hat\mu_i} - \ln\frac{y_i}{\hat\mu_i}\right\} .
\end{aligned}$$

Inverse Gaussian

$$\begin{aligned}
\Delta &= 2\sum_{i=1}^{n}\frac{y_i(-\frac{1}{2y_i^2}+\frac{1}{2\hat\mu_i^2}) + \frac{1}{y_i} - \frac{1}{\hat\mu_i}}{\sigma^2} \\
&= \frac{1}{\sigma^2}\sum_{i=1}^{n}\frac{y_i^2 - 2y_i\hat\mu_i + \hat\mu_i^2}{\hat\mu_i^2 y_i} \\
&= \frac{1}{\sigma^2}\sum_{i=1}^{n}\frac{(y_i-\hat\mu_i)^2}{\hat\mu_i^2 y_i} .
\end{aligned}$$

Negative binomial

$$\begin{aligned}
\Delta &= 2\sum_{i=1}^{n}\left\{y_i\left(\ln\frac{\kappa y_i}{1+\kappa y_i} - \ln\frac{\kappa\hat\mu_i}{1+\kappa\hat\mu_i}\right) - \frac{1}{\kappa}\ln(1+\kappa y_i) + \frac{1}{\kappa}\ln(1+\kappa\hat\mu_i)\right\} \\
&= 2\sum_{i=1}^{n}\left\{y_i\ln\frac{\kappa y_i}{\kappa\hat\mu_i} - y_i\ln\frac{1+\kappa y_i}{1+\kappa\hat\mu_i} - \frac{1}{\kappa}\ln\frac{1+\kappa y_i}{1+\kappa\hat\mu_i}\right\} \\
&= 2\sum_{i=1}^{n}\left\{y_i\ln\frac{y_i}{\hat\mu_i} - (y_i+1/\kappa)\ln\frac{y_i+1/\kappa}{\hat\mu_i+1/\kappa}\right\} .
\end{aligned}$$

5.4 *Show that the Anscombe residual for the Poisson distribution is*

$$\frac{3\left(y_i^{2/3} - \hat y_i^{2/3}\right)}{2\,\hat y_i^{1/6}} .$$

For the Poisson, $V(\mu) = \mu$. Therefore $h(y)$ is chosen such that

$$h(y) = \int \mu^{-1/3}\, d\mu = \frac{3}{2}\, y^{2/3} .$$

Also, $\dot{h}(\hat{y}_i) = \hat{y}_i^{-1/3}$ and $\sqrt{V(\hat{y}_i)} = \hat{y}_i^{1/2}$. This gives the Anscombe residual as

$$\frac{\frac{3}{2}\, y_i^{2/3} - \frac{3}{2}\, \hat{y}_i^{2/3}}{\hat{y}_i^{-1/3}\hat{y}_i^{1/2}} = \frac{3\left(y_i^{2/3} - \hat{y}_i^{2/3}\right)}{2\hat{y}_i^{1/6}} \; .$$