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An Old Tool - Modern Applications

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AN OLD TOOL - MODERN APPLICATIONS

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Summary

The mortality of a modern developed population is largely the mortality of old age, and most of the information about it may be summarised by two parameters, the first indicating the age near which most of the deaths occur and the second the spread of the deaths. The Gompertz “law” of mortality, so popular in years gone by, still provides a good general representation of the age pattern of mortality at these ages, and its parameters, the modal age at death and ageing parameter (equal to the force of mortality at the modal age) provide direct measures of location and spread.

The two Gompertz parameters are readily calculated if the upper and lower quartiles of the curve of deaths are know, and accurate approximations to numerous life table functions (e.g. e_x) are then immediately available.

Using only a life expectancy from a base period life table, the parameters of the approximating Gompertz distribution, and two applications of a straightforward formula, one can obtain very accurate approximations to the generational expectation of life at any age for any population in a constant uniform mortality change environment without the need to derive a special generational life table relating to the age and year of birth of the life involved.

1 Introduction

The mortality of a modern developed society is largely the mortality of old age, and most of the information about it may be summarised by two parameters, the first indicating the age near which most of the deaths occur, and the second indicating the spread of the deaths.

The broad shape of most, if not all, modern human life tables is still that of Gompertz (1825) although precise age-by-age fits are not usually obtainable. The Gompertz "law" also appears to have wide application among other animal species (Carey,1997; Olshansky and Carnes, 1997). Under this "law", the force of mortality (or hazard function) at age x may be written in the form

$$\mu_x = B e^{kx} . \quad (1)$$

The constant B indicates the base level of mortality in the life table, whilst the second constant k records the increase in mortality with age. The force of mortality at the mode of the density function $l_x \mu_x$ of the distribution of time to death (sometimes referred to as the *curve of deaths*) coincides with the Gompertz ageing parameter k, and formula (1) may therefore be re-written as

$$\mu_x = k \exp\{(x-m)k\}, \quad (2)$$

with m the mode of the curve of deaths.

Pollard and Valkovics (1992) investigated the Gompertz distribution with hazard function (1) over the entire range $(-\infty, \infty)$ and found that the standard deviation of the distribution was $(\pi/\sqrt{6})/k$. Since very little of the density of this distribution over $(-\infty, \infty)$ lies over the negative part of the axis for values of B and k encountered with human life tables, the standard deviation of the time to death under the Gompertz "law" of a life aged zero is close to $(\pi\sqrt{6})/k$. The mode m and ageing parameter k of the Gompertz "law" (2) are therefore the two essential parameters summarising modern mortality, providing measures of location and spread respectively.

How well do the parameters m and k summarise the overall pattern of mortality in a modern developed population? Before answering this question, we first recall certain results which appeared in the literature in recent years. Then, in section 3, we compare the answers obtained using these Gompertz-based formulae with exact calculations using certain national life tables, and find that the Gompertz approximations are generally very good.

The remainder of the paper is devoted to practical applications of the formulae and extensions which allow the demographer to use a simple cross sectional life table to obtain life expectancies and other life table functions in respect of a cohort of lives in a changing mortality environment.

2. Life table formulae under the Gompertz "law"

Exact formulae for the survivorship function l_x and the mortality rate q_x under the Gompertz "law" have been known since last century, and may be found in many standard texts (eg Jordan, 1967; Neill 1977). Explicit formulae for life expectancies, standard deviations of time to death, and other more complicated life table functions, on the other hand, were not available until comparatively recently, and it is these formulae which remove the necessity for tedious calculation.

Pollard and Valkovics (1992) obtained the cumulant function, exact moments and percentiles for the Gompertz distribution with hazard function (1) over the range $(-\infty, \infty)$. Their moment and percentile formulae may be re-arranged as follows in terms of the mode m and ageing constant k:

$$\text{mean} = m - \gamma / k = m - 0.5772157 / k \; ; \tag{3}$$

$$\text{standard deviation} = (\pi/\sqrt{6}) / k = 1.2825498 / k \; ; \tag{4}$$

$$100p \text{ percentile} = m + [\ln(-\ln(1-p))] / k \; ; \tag{5}$$

(γ is the Euler constant). Comparison of (3) and (5) reveals that the mean should correspond to the 43.0% point of the distribution, whilst (5) reveals that the mode must lie at the 63.2% point of the distribution.

The distribution for which equations (3)-(5) are exact lies over the range $(-\infty, \infty)$. For values of the parameters m and k typical of modern human mortality, however, very little of the density lies above the negative axis. Formulae (3)-(5) should therefore provide accurate approximations for the expectation of life at birth, the standard deviation of the time to death from birth, and the percentiles of the age at death random variable. The fact that the Gompertz distribution over $(-\infty, \infty)$ has a very small density on the negative part of the axis will compensate to some extent for the higher-than-Gompertz mortality in a normal life table below age 30.

In the case of a life table with radix l_0 = 100,000, we conclude from (3) and (5) that the survivorship function l_x should be close to 57,038 at an age equal to the expectation of life at birth, and from (5) that l_x should be close to 36,788 at the mode of the curve of deaths. Replacing 1-p by l_x/100,000 in (5), we can also deduce that

$$l_x \approx 100{,}000 \exp\{-\exp[k(x-m)]\} \; . \tag{6}$$

Pollard (1991) showed that in respect of a Gompertz life aged x, the following was true of the time T to death:

$$E(e^{kT}) = 1 + k / \mu_x \; ; \tag{7}$$

$$\text{standard deviation of } e^{kT} = k / \mu_x \; . \tag{8}$$

$$100p \text{ percentile of T distribution} = \ln[1 - e^{k(m-x)} \ln(1-p)] . \quad (9)$$

Using the standard statistical formulae for the expectation and variance of a random variable under a non-linear transformation, he was able to deduce from (7) and (8) formulae for the expectation and standard deviation of T, which may be re-arranged as:

$$e_x = E(T) \approx [\ln(1 + e^{k(m-x)}) - {}_{1/2} (1 + e^{k(x-m)})^{-2}] / k ; \quad (10)$$

$$\text{standard deviation of T} \approx [k(1 + e^{k(x-m)})]^{-1} . \quad (11)$$

An exact formula for the expectation of life at age x under the Gompertz mortality regime (2) can in fact be obtained by direct integration of the survivorship proportion ${}_tp_x$ from t=0 to t=∞:

$$e_x = \exp[e^{k(x-m)}] [m - x - \gamma/k + \psi(e^{k(x-m)})/k] \quad (12)$$

where

$$\psi(z) = z - z^2/(2\times 2!) + z^3/(3\times 3!) - z^4/(4\times 4!) + \dots . \quad (13)$$

Several points need to be noted at this stage. First, when x=0, $e^{k(x-m)}$ is very close to zero, and as a result, (12) effectively coincides with (3). Second, as x increases from 0, e_x declines in an approximately linear manner over much of the earlier part of the life span. The third observation is that $\psi(1) = 0.7965996...$, and as a result, the complete expectation of life at the modal age m is given by the exact formula

$$e_m = (0.5963473...)/ k , \quad (14)$$

a result which may prove useful when completing lifetables where the data at the older ages are sparse.

The series (13) is straightforward, but tedious to evaluate each time an expectation of life is required. A table of values of $\Psi(z)$ might be provided. However, for values of $z \leq 2$ (which covers all the life span from age zero to m + (ln 2)/k or approximately seven years beyond the mode), an accurate representation of $\psi(z)$ is provided by the empirical function

$$\xi(z) = z (1+0.227z)^{-1.111} . \quad (15)$$

For all ages up to about m + (ln 2)/k, therefore, we can use the approximate formula

$$e_x \approx \exp(z) [m - x - \gamma/k + (z/k)(1 + 0.227 z)^{-1.111}] , \quad (16)$$

with

$$z = e^{k(x-m)} . \quad (17)$$

The accuracy of formula (16) is evident from table (1), which shows for two Gompertz life tables (reflecting respectively male and female mortality in a modern developed population), the exact expectation of life and the formula (16) value at selected ages throughout the life span. For all ages up to m + (ln 2)/k, the formula produces values for the complete expectation of life which are virtually indistinguishable from the exact values. At older ages, the approximation (16) fails badly but formula (10) continues to produce reasonably reliable values.

3. Gompertz and modern mortality data

How well does the Gompertz "law" represent modern mortality data? To find an answer to this question, the mode and force of mortality at the mode were obtained for six modern life tables: the Australian Life Tables (males and females) 1990-92, English Life Table Number 14 (males and females), and male and female life tables of the Federal Republic of Germany for 1982-84. The mode of the curve of deaths occurs almost exactly half a year of age above the maximum point of the life table function d_x. By visual interpolation it is possible therefore to write down an approximation to it. For the more accurate purposes of this paper, the maximum point of the d_x was found by fitting a quadratic through the three d_x values straddling the largest d_x entry and maximizing the quadratic.The force of mortality at the mode (the life table ageing parameter) was then found by log-linear interpolation in the life table μ_x column. The mode and ageing parameter of each of the above life tables are exhibited across the first two rows of table 2.

The two German life tables have been calculated directly from the underlying population mortality data and are not graduated. As a result, they are not particularly smooth in the neighbourhood of the mode, and differentiation to determine the location of the mode is unreliable. To determine the mode, therefore, a modified (smooth) life table over the ten-year age range of interest was obtained by applying three- and five-term running averages to the tabled ln(q_x) values for the females and males respectively. The mode m and ageing parameter k were then obtained in the same way as for the other graduated tables.

Using only the mode m and ageing parameter k, estimates were prepared for the median, the lower quartile, the upper quartile, and the complete expectation of life at decennial ages under the assumption that mortality followed the Gompertz pattern. The formulae for the median and quartiles were the exact formulae for these measures in respect of the Gompertz distribution over (-∞, ∞) obtained by setting p=0.5, 0.25 and 0.75 in equation (5), and are approximate therefore for the assumed Gompertz life table. Formula (16) was used to approximate the the expectation of life values.

For each of the six life tables, the Gompertz approximations to the median, quartiles and life expectancies are shown in table 2 alongside the exact values obtained directly from the original life table (the ungraduated life table in the German cases). Based on the Gompertz distribution over (-∞, ∞), we would expect 57% of births to survive to an age equal to the complete expectation of life and 37% to survive to the modal age (section 2). Theoretical and life table values for these proportions are also shown in table 2.

The comparisons are generally very good, and suggest that the Gompertz model should provide reliable approximations over broad sections of the life table encompassing the prime mortality ages. At the same time, it should be noted that the model is unlikely to be particularly helpful over limited age ranges, particularly those well away from the centre of the curve of deaths.

4. The US male population 1953-1993

If an attempt is made to use the results of section 3 to compare the mortalities of a number of populations or study the change in mortality of a particular population over time, two problems become apparent immediately. First, the life tables of interest may have been graduated using different methods or may even be ungraduated; either way, determination of the Gompertz mode becomes very uncertain. We saw this already in the case of the German Federal Republic life table. Second, certain "modern developed populations" (e.g. the US male population 1953) still retain large infant mortality rates, and these distort the the efficacy of the section 3 approach.

How then can one obtain a representative Gompertz fit of a given life table with a minimum of effort, avoiding the above two problems? One very easy approach is to obtain the two quartile ages $x_{0.25}$ and $x_{0.75}$ of the survivors from age 1 using the l_x column of the life table. According to (5), the ageing parameter and mode of the fitted Gompertz distribution are then

$$k = 1.5725336 / (x_{0.75} - x_{0.25}) \; ; \qquad (18)$$

$$m = 0.207712\, x_{0.25} + (1 - 0.207712)\, x_{0.75} \; . \qquad (19)$$

If the infant deaths are all assumed to take place very close to age zero, then the expectation of life at age zero can be obtained by multiplying the expectation of life for age zero, as determined by equation (16), by l_1/l_0.

The approach described in the previous paragraph was adopted to obtain Gompertz fits to the lifetables of the US male population at five-yearly intervals from 1953 to 1993 inclusive. The life tables themselves were obtained using the World Health Organisation (WHO) mortality data for the USA and the LIFETIME software package, which uses interpolation between pivotal μ_x values at the centres of five-year age groups to compute the full life table. As a result there is a very small amount of graduation, but the discontinuities at the pivots make differentiation to determine the mode unreliable.

In terms of life expectancies, the results are pleasing (table 3). The trend of improving life expectancy is clearly evident from the movement of the mode m. Apart from the setback of the 1960s, the mode has moved relentlessly forward and from 1978 onwards appears to have forgotten its 1960s downwards detour and resumed its earlier trajectory (figure 1). The variance of the time to death, $\pi^2/(6k^2)$ has generally declined over time (figure 2).

Of the nine tables fitted, 1993 is perhaps the worst, and the reason for this appears to be the relatively high level of youth and young adult mortality which persists, and is not accounted for in the Gompertz model. In an attempt to remedy this, a Makeham modification of the Gompertz "law" (Makeham, 1860) was studied as an alternative model, but the improvements in fit for the expectation of life were relatively minor and involved rather more complicated calculations. They also led to negative values for the additive constant for some of the earlier US male life tables. The simpler Gompertz approach was therefore retained.

The broad representation of the 1993 US male mortality by the three parameter Gompertz approach is evident in figure 3. It is clear that the model cannot be used to estimate mortality at individual ages.

Formulae equivalent to (18) and (19) are easily derived for situations where the percentiles are not the quartiles. If the two percentile ages are respectively x and y, then

$$k = N / (x-y) \; ; \tag{20}$$

$$m = W x + (1-W) y \; ; \tag{21}$$

with

$$N = \ln[\ln(1-l_x/l_1)/\ln(1-l_y/l_1)] \; ; \tag{22}$$

$$W = \ln[-\ln(1-l_x/l_1)] \; . \tag{23}$$

5. Period and generational lifetables under constant uniform changes in mortality

Whilst no modern life table follows the Gompertz "law" closely, we noted earlier that the broad shape of most if not all life tables is still essentially of this form. The model allows useful accurate approximations of many functions based on the life table (Pollard, 1991). The relationship between period and generation life tables under constant uniform mortality change is another area in which such approximations can be made, and the methods underlying these calculations depend on the following four important observations concerning a life table of the Gompertz form.

(a) The force of mortality at the mode of the curve of deaths is equal to k.

(b) The effect on the life table of multiplying the force of mortality at every point throughout the lifespan by the same factor r^N (where r, the annual mortality improvement factor, is typically in the neighbourhood of 1.0) is to produce a new Gompertz life table in which a life aged x has mortality equivalent to a life in the original table aged $x + N \ln(r)/k$.

(c) If a population has a cross sectional (period) life table of the Gompertz form with ageing parameter k, and all its members are subject to a constant on-going annual improvement factor r at every age, then a member of the population aged x with force of mortality μ_x under the period table has a Gompertz generational life table with ageing parameter $k+\ln(r)$.

(d) If the force of mortality at exact age x is fixed at μ_x and the rate of increase in mortality represented by k is allowed to change, then the change in the complete expectation of life at age x corresponding to the change in k is given by the derivative formula

$$d(e_x)/dk = [1 - (\mu_x + k) e_x]/k^2 . \qquad (24)$$

The second derivative can be calculated from the first by means of the formula

$$d^2(e_x)/dk^2 = -[(\mu_x+3k) d(e_x)/dk + e_x]/k^2 . \qquad (25)$$

Observations (a) and (b) are well-known and are easily proved. Observation (c) is also easily proved. Proofs of formulae (24) and (25) are given in the appendix. The same methods also provide a formula for the derivative of a life annuity (expected discounted future years of life), which may find application in stable population theory (Pollard, 1998).

6. Period and generational life expectancies under constant uniform mortality change

The demographer is often asked the expectation of life of a member of the community. A common approach to finding an answer is to consult the most recent national life table and quote the complete expectation of life corresponding to the enquirer's age. Assuming that the individual is in average health, the answer obtained in this manner is usually conservative, for two reasons: first, the national life table is based on data some years prior to the enquiry and is therefore out of date, and second, the approach uses period data and makes no allowance for further improvements in mortality during the individual's lifetime.

It is of course possible to make assumptions about mortality trends and construct a special generational life table for the enquirer and so deduce his/her life expectancy. Such a process is computationally straightforward, but somewhat of a nuisance for a single enquiry. Furthermore, different generational tables have to be produced for different ages of enquirers and for different assumptions about mortality improvement. The observations of section 5 allow the demographer to produce an accurate approximation to the generational life expectancy after only a few lines of arithmetic.

To demonstrate the approach, consider a US male aged 50 in 1998 and in average health. The most recent available life table is the US male life table 1993 referred to earlier, which is about five years out of date at the time the life expectancy of the 50-year-old male is required.

Over the decade to 1993, US male mortality in the important age range 75-95 improved by about 1% per annum. The force of mortality at the modal age of the 1993 life table is 0.08164 (table 3). If the rate of mortality improvement over the decade to 1993 has continued in subsequent years, then, under a Gompertz approximation and observation (b), one can treat the male aged 50 in 1998 as equivalent to one aged

$$50 + 5 \ln(0.99)/0.08164 = 49.38$$

in 1993. The life expectancy of a 49.38-year-old male according to the 1993 life table is 27.23. This will be the life expectancy for a male aged 50 under the synthesised 1998 *period* life table.

We now need to determine the life expectancy of the 50-year old under the generational life table pertaining to males aged 50 in 1998. To do this, we note that $\mu_{49.38}$=0.00568. According to formulae (24) and (25), therefore, the first two derivatives of the complete expectation of life are

$$d(e_{49.38})/dk \ = \ [1 - (0.00568 + 0.08164)\times 27.23]/(0.08164)^2 \ = \ -207 \ ;$$

$$d^2(e_{49.38})/dk^2 \ = \ -[(0.00568+3\times 0.08164)\times(-207) + 27.23]/(0.08164)^2 \ = \ 3{,}665.$$

Invoking observation (c) of section 5 and applying the Taylor series expansion, the generational life expectancy of the male aged 50 in 1997 will be

$$\begin{aligned} e_{50(1998)} &= 27.23 + (-207)\ln(0.99) + {\scriptstyle 1/2}\,(3{,}665)\,[\ln(.99)]^2 \\ &= 29.48\ . \end{aligned}$$

Full computation of a special generational life table for a male aged 50 in 1998 under the given mortality improvement assumptions yields an exact generational life expectancy of 29.48, i.e. exactly the same answer to two decimal places.

Comparisons for other 1998 ages under the same scenario are shown in table 4. The effects of including only the linear term and including both linear and quadratic terms are shown in this table, and it is clear that the quadratic Gompertz-based method produces accurate approximations at all ages. Reasonable approximations are obtained using the linear term alone from about age 50 onwards, and might be expected to be obtained at even younger ages in situations where the annual rate of mortality change is smaller. For most purposes, therefore, only the first derivative is required. Calculations under different mortality improvement assumptions indicate that the discrepencies between the Gompertz approximations and the exact calculations are minor compared with the the effects of small changes in the very uncertain mortality improvement assumption.

Another approach based on (16) rather than (24) and (25) is also possible. This is based on the observation that for a population having a cross-sectional Gompertz life table with parameters k and m, under constant uniform mortality change with annual improvement factor r, the generational life table for a person aged x in the period year is Gompertz with ageing parameter k^*=k+ln(r) and mode

$$m_x^* \ = \ x + [\ln(k^*/k) - k(x-m)]/k^* \ . \qquad (26)$$

All that one needs to do therefore to obtain the generational life expectancy of a life is to calculate the period life expectancy as previously and then add the difference between the Gompertz period and generational life expectancies, both calculated using (16). The results obtained using this approach are set out in the penultimate column of table 4, and it is clear that they reflect the exact values very closely and are generally superior to those obtained by the Taylor series method. The calculations are also simpler.

All the generational life tables used for comparative purposes have been computed assuming that the mortality improvement factors apply to the q mortality rates rather than the force of mortality, as assumed in the theoretical development of the approximate formulae.

Calculations designed to compare the effects of the two different approaches indicate that the distinction between them is negligible in its effect except at extreme old age.

The force of mortality in any real life table is high immediately after birth and falls away rapidly; it is usually considered indeterminate at age 0. By contrast, under the Gompertz "law", the force of mortality continues to decline with reducing age. Under modern low mortality regimes, it is the mortality at the adult ages and improvements in this mortality which have the major impact on life expectancy. For the application of the Taylor series method, therefore, the high, indeterminate μ_0 value is not used in evaluating the derivatives, but μ_1.

The adjustment for mortality improvement between 1993 and 1998 for a male aged 0 in 1999 leads to consideration of a life aged between -1 and 0 in the 1993 table. In our interpolation calculations, the expectation of life of a male aged -1 has been treated as being one year greater than the 1993 expectation of life at birth (ie zero mortality before birth).

7. A further example

The following information only is provided in respect of the US female population in 1993: l_0=100,000, l_1=99,258, l_{72}=75,434, l_{73}=73,602, l_{89}=27,482 and l_{90}=24,298. What is the expectation of life for a female aged 65 in 1998 if mortality has improved at an average of about 1.25% per annum since 1993 and is expected to improve at 1% per annum from 1998 onwards?

The necessary calculations are as follows:

$x_{0.25}$ = 72.54 [linear interpolation after adjustment for q_0]
$x_{0.75}$ = 89.79 [linear interpolation after adjustment for q_0]
k = 0.10002 [formula (18)]
m = 86.21 [formula (19)]
equivalent 1993 age = 65+5 ln(0.9875)/0.10002 = 64.37 [observation (b)]
k^*=0.10002+ln(0.99)=0.08997
$m_{64.37}{}^*$=87.47 [formula (26)]
$e_{65(1998)}$=20.44 [formula (16)]

Computation of a special generational life table for a female aged 65 in 1998 produces an exact answer of 20.92. The method we have used is equivalent to using a period life expectancy for age 64.37 of 19.21 (using k and m in formula (16)). If the actual 1993 life table life expectancy value for age 64.37 had been used (19.55), and the difference approach of the previous section adopted, an estimated generational life expectancy of 20.78 would have emerged. It is clearly preferable to use known life expectancies for the period life table whenever possible.

The expectation of life for age 65 shown in the original 1993 period table is 19.07; so the effect of improving mortality is to increase life expectancy by 1.85 years.

8. Concluding remarks

With mortality at the infant, childhood and young adult ages now so low in most modern developed populations, life expectancy at any age is determined largely by mortality at the older ages, most of the information about which can be summarised by two parameters, one providing an indication of the age near which most deaths occur and the other indicating the spread of the ages at death. Although other more detailed models may provide more accurate representations of mortality age by age, the Gompertz "law" still provides the simplest most accurate representation of old age mortality, and its parameters, the mode and force of mortality at the mode, provide respectively the measures of location and spread. Since infant mortality is still appreciably higher than for any age prior to the senescent years, a third parameter q_0 needs to be employed, if mortality and expectation of life are to be summarised reasonably adequately over the whole life span. Indeed, a strong case could be made for a set of model life tables applicable to modern developed populations based on these three parameters. In adopting such an approach one would be omitting from the model the largely avoidable but not insignificant mortality of the younger adult ages, and for certain purposes such an omission would be unacceptable, but for others it would be of little consequence.

The Gompertz "law" of mortality can also be a very useful model for obtaining quick accurate approximations for many life table functions. In particular, it allows the rapid calculation of life expectancies in an environment of mortality change, and avoids the need to prepare special generational life tables for different cohorts of lives and different assumptions about future mortality change. All that is required is a period life expectancy from the base life table and two applications of (16).

More and more persons are living to 80 and beyond. Almost 60 per cent of US females, for example, live beyond that age according the the 1993 life table, and close to 40 per cent of US males. The numbers surviving 85 are also high: more than 40 per cent of US females and around 25 per cent of US males. It is surprising therefore that many life tables (which provide the distribution of the age at death) still use a final open-ended age interval from an age as low as 85 or even 80. The user has to conjecture about a very large part of the distribution of the time to death, given only the earlier part of it. Indeed, if the open interval commences as 80 and the population is female, the user has to guess more than half the distribution! The Gompertz formulae we have discussed will be of help to the user interested in estimating the missing part of the distribution.

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Appendix. The effect on the expectation of life of changing the ageing parameter in the Gompertz "law"

Consider the function

$$Z \;=\; -\ln {}_tp_x \;=\; \int_0^t \mu_{x+u}\,du \;. \qquad \text{(A1)}$$

Under Gompertz mortality, this function becomes

$$Z \;=\; \mu_x \int_0^t e^{ku}\,du \;. \qquad \text{(A2)}$$

For given fixed μ_x but varying senescence parameter k therefore

$$dZ/dk \;=\; \mu_x \int_0^t u\,e^{ku}\,du \;, \qquad \text{(A3)}$$

which on integration and rearrangement yields

$$dZ/dk \;=\; (k\,t\,\mu_{x+t} - \mu_{x+t} + \mu_x)/k^2 \,. \qquad \text{(A4)}$$

We now note that since ${}_tp_x = e^{-Z}$,

$$d({}_tp_x)/dk \;=\; -\,{}_tp_x\,dZ/dk \,. \qquad \text{(A5)}$$

Under the Gompertz "law", therefore,

$$d({}_tp_x)/dk \;=\; (-k\,t\,{}_tp_x\,\mu_{x+t} + {}_tp_x\,\mu_{x+t} - \mu_x\,{}_tp_x)/k^2 \;. \qquad \text{(A6)}$$

Integration of (A6) with respect to t from 0 to ∞ yields

$$d(e_x)/dk \;=\; [1 - (\mu_x + k)\,e_x]/k^2 \,. \qquad \text{(A7)}$$

Multiplication of (A7) by k^2 and differentiation with respect to k (keeping μ_x constant) leads to

$$d^2(e_x)/dk^2 \;=\; -[(\mu_x+3k)\,d(e_x)/dk \;+ e_x] \,/\, k^2 \;, \qquad \text{(A8)}$$

a process which may be repeated to obtain higher-order derivatives.

Table 1
Comparison of the expectations of life computed using (16) and (10) with the exact values under the Gompertz "law"

	"Male population" with k=0.09334 m=81.03			"Female population" with k=0.10810 m=86.73		
Age x	**Exact e_x**	**Equation (16)**	**Equation (10)**	**Exact e_x**	**Equation (16)**	**Equation (10)**
0	74.89	74.89		81.40	81.40	
10	64.95	64.95		71.41	71.41	
20	55.07	55.07		61.44	61.44	
30	45.32	45.32		51.52	51.52	
40	35.85	35.85		41.72	41.72	
50	26.87	26.87		32.17	32.17	
60	18.76	18.76		23.15	23.15	
70	11.94	11.94		15.13	15.13	
80	6.81	6.82		8.71	8.71	
85	4.93	4.95		6.25	6.25	
86	4.60	4.63		5.82	5.82	
87	4.29	4.32		5.41	5.41	
88	4.00	4.03 (a)	3.87	5.02	5.03	
89	3.72	3.73	3.61	4.65	4.66	
90	3.46	3.41	3.36	4.31	4.32	
91	3.22	3.04	3.13	3.98	4.00	
92	2.98		2.91	3.67	3.70	
93	2.77		2.71	3.38	3.40 (a)	3.27
94	2.56		2.51	3.11	3.10	3.02
95	2.37		2.33	2.86	2.76	2.78
100	1.59		1.57	1.83		1.81
105	1.04		1.04	1.14		1.13
110	0.67		0.67	0.69		0.69

(a) For the "males" the effective range for (16) terminates near age m+(ln 2)/k = 88; for females the effective range terminates near age m + (ln 2)/k = 93. Beyond these ages, formula (10) should be used.

Table 2
Life table functions based on a two parameter Gompertz model compared with functions calculated exactly in the source life table

Life table function	Aust. 90/92 Males		Aust. 90/92 Females		ELT 14 Males		ELT 14 Females		FRG Males 82/84		FRG Females 82/84	
mode m	81.03		86.73		77.23		84.33		77.69		83.82	
k	0.09334		0.10810		0.09015		0.11154		0.09180		0.11257	
	G	L	G	L	G	L	G	L	G	L	G	L
median	77.1	77.4	83.3	83.3	73.2	73.8	81.0	80.2	73.7	74.2	80.6	80.7
lower q'tile	67.7	68.0	75.2	74.8	63.4	64.6	73.2	71.0	64.1	64.3	72.8	72.4
upper q'tile	84.5	84.7	89.8	89.8	80.9	81.2	87.3	86.9	81.2	81.5	86.7	86.8
e_{0}	74.9	74.3	81.4	80.4	70.9	71.0	79.2	77.0	71.5	70.8	78.7	77.5
e_{10}	65.0	65.1	71.4	71.1	61.0	62.2	69.2	68.0	61.6	61.9	68.7	68.4
e_{20}	55.1	55.4	61.4	61.2	51.2	52.5	59.2	58.1	51.7	52.3	58.7	58.6
e_{30}	45.3	46.1	51.5	51.5	41.6	42.9	49.3	48.3	42.1	42.8	48.8	48.8
e_{40}	35.9	36.7	41.7	41.8	32.3	33.3	39.5	38.7	32.8	33.5	39.0	39.2
e_{50}	26.9	27.5	32.2	32.3	23.7	24.3	30.0	29.4	24.1	24.6	29.5	29.9
e_{60}	18.8	19.1	23.2	23.4	16.1	16.4	21.1	20.9	16.4	16.8	20.7	21.2
e_{70}	11.9	12.1	15.1	15.4	10.0	10.1	13.3	13.4	10.1	10.3	12.9	13.3
e_{80}	6.8	7.0	8.7	8.9	5.6	5.8	7.3	7.5	5.6	5.8	7.0	7.2
e_{90}	3.4	3.9	4.3	4.6	2.7	3.3	3.4	3.9	2.7	3.5	3.2	3.7
${}_{e}p_{0}$	0.57	0.60	0.57	0.60	0.57	0.59	0.57	0.60	0.57	0.60	0.57	0.62
${}_{m}p_{0}$	0.37	0.37	0.37	0.38	0.37	0.38	0.37	0.34	0.37	0.38	0.37	0.37

Notes: G = Gompertz approximation.
L = exact life table value.
${}_{e}p_{0}$ = proportion surviving from birth to an age equal to the complete expectation of life at birth.

Table 3
US male mortality 19953-1993: life expectancies according to the original life table and the Gompertz "law"

Age	**Expectation of life in**											
	1953		**1959**		**1963**		**1968**		**1973**		**1978**	
	LT	**G**	**LT**	**G**	**LT**	**G**	**LT**	**G**	**LT**	**G**	**LT**	**G**
0	66.0	66.4	66.6	67.0	66.6	66.9	66.6	66.9	67.5	67.9	69.6	70.0
1	67.1	67.5	67.7	68.1	67.5	68.0	67.2	67.6	67.9	68.3	69.7	70.1
10	58.7	58.7	59.2	59.2	59.0	59.1	58.6	58.8	59.3	59.5	61.1	61.3
20	49.3	49.1	49.7	49.6	49.5	49.4	49.2	49.2	49.9	49.8	51.6	51.6
30	40.2	39.8	40.5	40.2	40.3	40.1	40.2	39.8	40.9	40.4	42.5	42.1
40	31.1	31.0	31.4	31.3	31.2	31.1	31.1	30.9	31.8	31.5	33.3	33.0
50	22.9	22.9	23.0	23.1	22.8	22.9	22.8	22.8	23.4	23.3	24.8	24.6
60	15.9	15.9	15.9	16.0	15.7	15.8	15.7	15.7	16.2	16.1	17.3	17.2
70	10.4	10.3	10.4	10.3	10.2	10.1	10.2	10.0	10.5	10.3	11.3	11.2
80	6.2	6.1	6.1	6.1	6.0	5.9	6.3	5.9	6.4	6.1	6.9	6.7
90	3.9	3.3	3.5	3.2	3.1	3.1	3.4	3.1	3.7	3.2	4.1	3.5
k	0.07732		0.07962		0.08029		0.07967		0.08014		0.08074	
m	75.72		76.12		75.94		75.67		76.33		78.11	

Age	**Expectation of life in**					
	1983		**1988**		**1993**	
	LT	**G**	**LT**	**G**	**LT**	**G**
0	70.9	71.3	71.5	72.0	72.1	72.7
1	70.8	71.2	71.3	71.8	71.8	72.4
10	62.1	62.3	62.5	62.9	63.0	63.5
20	52.5	52.5	53.0	53.1	53.5	53.8
30	43.3	43.0	43.9	43.6	44.3	44.2
40	34.1	33.8	34.8	34.4	35.4	35.0
50	25.4	25.2	26.1	25.8	26.7	26.4
60	17.7	17.6	18.3	18.2	18.8	18.7
70	11.5	11.4	11.9	11.8	12.3	12.3
80	6.9	6.7	7.0	7.1	7.2	7.5
90	4.0	3.5	4.0	3.7	3.8	4.0
k	0.08391		0.08279		0.08164	
m	78.96		79.63		80.36	

Table 4
Comparison of expectations of life

Age x	Complete expectation of life according to the following tables:				
	US males Life Table 1993	**Gompertz projected generational life table (linear term only)**	**Gompertz projected generational life table (linear and quadratic terms)**	**Alternative approach**	**Specially projected generational life tables**
0	72.11	80.21[a]	81.04[a]	81.30[a]	81.35
1	71.78	79.38	80.20	80.41	80.92
10	63.01	69.94	70.62	70.82	70.85
20	53.49	59.31	59.88	59.90	59.91
30	44.34	49.04	49.47	49.46	49.45
40	35.37	38.99	39.29	39.27	39.26
50	26.71	29.30	29.48	29.51	29.48
60	18.81	20.52	20.62	20.67	20.63
70	12.26	13.28	13.32	13.38	13.34
80	7.16	7.69	7.70	7.77	7.73
90	3.80	4.06	4.06	3.98	4.07

[a]See final two paragraphs of section 5.