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PDE Approach for Risk Measures for Derivatives With Regime Switching

In this talk, we shall discuss a partial differential equation (P.D.E.) approach to evaluate coherent risk measures for derivative securities in a Markovian regime-switching Black-Scholes-Merton environment. In such a paradigm, the dynamics of underlying risky asset are governed by a Markovian regime-switching Geometric Brownian Motion; that is, the appreciation rate and the volatility in the log-normal dynamics of the underlying risky asset switch over time according to the state of a continuous-time, finite-state, Markov chain. The states of the chain are interpreted as different states of an economy. The P.D.E. approach provides market practitioners with a flexible and effective way to evaluate risk measures in the Markovian regime-switching Black-Scholes-Merton model. We shall demonstrate the use of the P.D.E. approach for evaluating risk measures for complex options, such as American options and barrier options.

Joint work with Robert J. Elliott and Leunglung Chan.

A P.D.E. Approach for Risk Measures for Derivatives With Regime Switching

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§1. Background and Main Ideas

- The global financial crisis of 2008 and derivative securities

- Risk management for derivative securities

- Quantitative models ?

- Cultural revolution in the history of (quantitative) finance ?

- Combine expert opinion (qualitative information) and quantitative models (quantitative information), like Bayesian statistics

- Some bird-eye views on the issue:

  1. Speculative activities: Appropriate methods to measure the unhedged risks

  2. Structural changes in economic conditions

  3. A coherent and scientific combination of expert opinion and quantitative information

- Modeling issues for risk assessment and management

  1. Identify risk drivers and model their dynamics

  2. Probability measures: risk-neutral v.s. real-world

  3. Risk measures

  4. Method to evaluate the risk measures

- Limitations of existing theories and derivative risks

  1. Traditional theories: Linear Risk
  2. Bigger Universe of Nonlinear Risk: Not well-explored!
  3. **Derivative securities** and Hedge Funds: Nonlinear Risk Behavior
  4. Call for new theories and tools for nonlinear risks
  5. Current Practice: Traders use Greek Letters, such as Delta; Sensitivities
  6. Nonlinearity: Dynamics of Risk Factors; Nonlinear Dependence; Functional Relationships of Risk Factors

- Key points of our work:

  1. Develop an appropriate risk measurement paradigm for unhedged or speculative risks of derivative securities based on coherent risk measures first proposed by Artzner, Delbaen, Eber and Heath (1999)

  2. Consider a Markovian regime-switching framework for modeling asset price movements

  3. Provide a practical approach based on partial differential equations (P.D.E.s) to evaluate risk measures for derivatives

  4. Demonstrate the use of the proposed approach for evaluating risk measures of complex derivative securities

- Why we consider the P.D.E. approach?

  1. Connection between probability theory and P.D.E.s: Foundations of Probability Theory

  2. P.D.E.s: An important mathematical tool to model evolving systems in diverse fields

  3. Availability of numerical schemes and commercial packages

  4. Flexible to accommodate different products

  5. Taste and culture of practitioners and researchers

§2. The Markovian regime-switching paradigm for asset price dynamics

- Consider a financial model consisting of two primitive assets - a bank account B and a share S

- Consider a continuous-time, N-state observable Markov chain $\{\mathbf{X}(t)\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ whose states represent different states of an economy, where $\mathcal{P}$ is a reference probability measure

- For each $t \in [0, T]$, $\mathbf{X}(t)$ takes a value from the canonical state space $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0) \in \Re^N$.

- $\{\mathbf{A}(t)\}$: A family of rate matrices of the Markov chain $\mathbf{X}$, which specifies the probability law of $\mathbf{X}$

- Semi-martingale dynamics (see Elliott, Aggoun and Moore (1994)):

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}(u)\mathbf{X}(u-)du + \mathbf{M}(t) \ ,$$

  where $\{\mathbf{M}(t)\}$ is an $\Re^N$-valued, $(F^{\mathbf{X}}, \mathcal{P})$-martingale.

- Predictability of the bounded variation term which is not a local martingale => $\mathbf{X}$ is a special semi-martingale => Uniqueness of the semi-martingale decomposition

- The market parameters: Let r denote the constant market interest rate and

$$\mu(t) = \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle \ , \quad \sigma(t) = \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle \ ,$$

where $\boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_N)'$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_N)'$ with $r_i > 0$, $\mu_i, \sigma_i \in \Re$, for each $i = 1, 2, \ldots, N$.

- The price dynamics for B and S under $\mathcal{P}$:

$$\begin{aligned} dB(t) &= rB(t)dt \ , \quad B(0) = 1 \ , \\ dS(t) &= \mu(t)S(t)dt + \sigma(t)S(t)dW(t) \ , \\ S(0) &= s \ . \end{aligned}$$

- For $N = 2$, one country two systems in Hong Kong (Mr. Tang, Former Chairman of PRC)

- Historical Remarks:

  1. Early Works: Quandt (1958) and Goldfeld and Quandt (1973) on regime-switching regression models and their applications to modeling nonlinearity in economic data

  2. Early Development of Nonlinear Time Series Analysis: Tong (1977, 1978, 1980) on the SETAR models; Ideas of Probability Switching

  3. Economics and Econometrics: Hamilton (1989) on Markov-switching autoregressive time series models

  4. Finance: Niak (1993), Guo (2001), Buffington and Elliott (2001) and Elliott, Chan and Siu (2005)

  5. Actuarial Science: Hardy (2001) and Siu (2005)

- Question: Why we consider the Markovian regime-switching model?

  1. Explain some important empirical features of financial time series (i) the heavy-tailedness returns' distribution (Extreme Value Theory advocated by Professor P. Embrechts; Mixing effect of volatility) (ii) time-varying conditional volatility (iii) volatility clustering (expressed discontinuously; intensity matrix)

  2. Nonlinearity and non-stationarity (Long-term Risk Management)

  3. Structural changes in economic conditions; business cycles

  4. Describe the stochastic evolution of investment opportunity sets

§3. A two-step paradigm for risk measurement

- First step: Use a price kernel for marking the derivative position to the model (Black-Scholes-Merton world in 1970's)

- Second step: Use a family of real-world, or subjective, probabilities for evaluating the unhedged risk of the derivative position (Bachelier-Samuelson world in 1900's and 1960's)

- Why? Use a risk-neutral measure if the unhedged risk can be traded and the market is liquid

  Example: Insurance contracts; Price as if they were options provided that they are traded in a competitive market.

- Literature: Siu and Yang (2000), Siu, Tong and Yang (2001), Boyle, Siu and Yang (2002), Cisneros-Molina (2006) (PhD thesis, Oxford University) and Rebonato (2007) (The Plight of Fortune Tellers).

- Market incompleteness due to the regime-switching risk

- More than one price kernels for making to the model

- Gerber and Shiu (1994): Esscher transform for option valuation

- Esscher transform:

  1. Time-honor tool in actuarial science (Esscher (1932))

  2. Exponential tilting; Edgeworth expansion of Bootstrap

  3. Might be related to the S-transform in the White Noise Theory introduced by Professor T. Hida in 1975

- Many works focus on Lévy-based asset price models

- Specification of a price kernel by the regime-switching Esscher transform (Elliott, Chan and Siu (2005))

- The regime-switching Esscher transform:

1. Define a process $\theta := \{\theta(t)\}$ by:

$$\theta(t) = \langle \boldsymbol{\theta}, \mathbf{X}(t) \rangle \ ,$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)' \in \Re^N$.

2. The regime-switching Esscher transform $\mathcal{Q}_\theta \sim \mathcal{P}$ on $G(t) := F^{\mathbf{X}}(t) \vee F^W(t)$ associated with $\theta := \{\theta(t)\}$:

$$\left. \frac{d\mathcal{Q}_\theta}{d\mathcal{P}} \right|_{G(t)} := \frac{\exp(\int_0^t \theta(u) dW(u))}{\mathrm{E}[\exp(\int_0^t \theta(u) dW(u)) | F^{\mathbf{X}}(t)]} \ .$$

- Martingale condition => Find θ such that

$$\tilde{S}(u) = \mathsf{E}^{\theta}[\tilde{S}(t)|\mathcal{G}(u)] \ , \quad t \geq u \ ,$$

where $\mathsf{E}^{\theta}[\cdot]$ is expectation under $\mathcal{Q}^{\theta}$ and $\tilde{S}(t) := e^{-rt}S(t)$.

- Risk-neutralized process θ:

$$\theta(t) = \sum_{i=1}^{N} \left(\frac{r - \mu_i}{\sigma_i} \right) \langle \mathbf{X}(t), \mathbf{e}_i \rangle \ .$$

- Let $W^{\theta}(t) := W(t) - \int_0^t \theta(u)du$. Then, by Girsanov's theorem, $\{W^{\theta}(t)\}$ be a $(G, \mathcal{Q}^{\theta})$-B.M. Under $\mathcal{Q}^{\theta}$,

$$dS(t) = rS(t)dt + \sigma(t)S(t)dW^{\theta}(t) \ .$$

- Consistent with the Minimum Entropy Martingale Measure (MEMM)

- May be related to the Minimal Martingale Measure of Föllmer and Schweizer (1991) based on the orthogonal martingale representation in Elliott and Föllmer (1991)

- Consider an option with payoff $V(S(T))$ at maturity T

- Given $S(t) = s$ and $\mathbf{X}(t) = \mathbf{x}$, a conditional price of the option is given by:

$$V(t, s, \mathbf{x}) = \mathsf{E}^{\theta}[e^{-r(T-t)}V(S(T))|S(t) = s, \mathbf{X}(t) = \mathbf{x}] \ .$$

- **Proposition 1:** Let $V_i := V(t, s, \mathbf{e}_i)$, for each $i = 1, 2, \cdots, N$, and write $\mathbf{V} := (V_1, V_2, \cdots, V_N)' \in \Re^N$. Then, V_i, $i = 1, 2, \cdots, N$, satisfy the following system of N-coupled P.D.E.s:

$$-rV_i + \frac{\partial V_i}{\partial t} + rs\frac{\partial V_i}{\partial s} + \frac{1}{2}\sigma_i^2 s\frac{\partial^2 V_i}{\partial s^2} + \langle \mathbf{V}, \mathbf{A}(t)\mathbf{e}_i \rangle = 0 \ ,$$

with terminal conditions $V(T, s, \mathbf{e}_i) = V(S(T))$, $i = 1, 2, \cdots, N$.

- Useful when trading is thin or market quotes are not available

- Coherent risk measures (Artzner et al. (1999))

  1. A set of theoretical properties a risk measure should satisfy

  2. Subadditivity: Allocating risk over different assets reduces risk

  3. Value-at-Risk: Not Sub-additive => Not Coherent

  4. Representation form of a coherent risk measure: The supremum of expected future net loss over a set of probability measures (Generalized Scenario Expectation, GSE)

- Use of the representation form:

  1. Develop a general risk measurement framework

  2. Link to other risk measures, such as Expected Shortfall and the weighted premium (Venter (1991), Furman and Zitikis (2008) and Woo and de Jong (2009)); Choice of probability measures or generalized scenarios

  3. Apply the P.D.E. approach to these risk measures for derivatives

  4. Relate to scenario-based risk measures used in practice

  5. Incorporate subjective views or expert opinion

- Future net loss of the option position over $[t, t+h]$:

  $$\Delta V(t,h) := e^{rh} V(t, S(t), \mathbf{X}(t)) - V(t+h, S(t+h), \mathbf{X}(t+h)) \ .$$

- Generate a family of subjective probability measures, or generalized scenarios

- Subjective Probabilities: Bayesian analysis; Robustness analysis in economic theory; Stress testing and scenario analysis in financial risk management; Profit testing in actuarial science (Heriot-Watt)

- For each $i = 1, 2, \cdots, N$, let $\Lambda_i = [\lambda_i^-, \lambda_i^+]$. For example, when $N = 2$ (i.e. State 1 is "Good Economy" and State 2 is "Bad Economy"), $\lambda_1^- = 0.05$; $\lambda_1^+ = 0.10$; $\lambda_2^- = 0.01$; $\lambda_2^+ = 0.05$.

- Market, or expert, opinion; "Think about the worst and Act on the best (Quotation: Chairman Mao)"

- Group Discussion: Convergence of view and come up with a set of scenarios (Breakwell (2007))

- In practice, N can be taken to be "2" or "3" (Taylor (2005))

- Suppose $\lambda(t)$ denotes the subjective appreciation rate of the share at time t. The chain modulates $\lambda(t)$ as:

$$\lambda(t) = \langle \boldsymbol{\lambda}, \mathbf{X}(t) \rangle \ ,$$

where $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \cdots, \lambda_N)' \in \Re^N$ with $\lambda_i \in \Lambda_i$, $i = 1, 2, \cdots, N$.

- Write Θ for the space of all such processes $\lambda := \{\lambda(t)\}$.

- Consider, for each $\lambda \in \Theta$, a process $\{\theta^{\lambda}(t)\}$ defined by putting

$$\theta^{\lambda}(t) = \sum_{i=1}^{N} \left(\frac{\mu_i - \lambda_i}{\sigma_i} \right) \langle \mathbf{X}(t), \mathbf{e}_i \rangle \ .$$

- The regime-switching Esscher transform $\mathcal{P}_{\theta^\lambda} \sim \mathcal{P}$ on $G(t)$ with respect to $\{\theta^\lambda(t)\}$:

$$\left.\frac{d\mathcal{P}_{\theta^\lambda}}{d\mathcal{P}}\right|_{G(t)} := \frac{\exp(\int_0^t \theta^\lambda(u) dW(u))}{\mathrm{E}[\exp(\int_0^t \theta^\lambda(u) dW(u)) | F^{\mathbf{X}}(t)]} \ .$$

- Under $\mathcal{P}_{\theta^\lambda}$,

$$dS(t) = \lambda(t) S(t) dt + \sigma(t) S(t) dW^\lambda(t) \ ,$$

where $\{W^\lambda(t)\}$ is a $(G, \mathcal{P}_{\theta^\lambda})$-standard B.M.

- Given $S(u) = s$ and $\mathbf{X}(u) = \mathbf{x}$, $u \in [t, t+h]$, the generalized scenario expectation for the option position V over $[t, t+h]$:

$$\begin{aligned} & \rho(u, s, \mathbf{x}) \\ := \; & \sup_{\lambda \in \Theta} \mathsf{E}^{\theta^{\lambda}}[\exp(-r(t+h-u))\Delta V(t,h) | S(u) = s, \mathbf{X}(u) = \mathbf{x}] \;, \end{aligned}$$

where $\mathsf{E}^{\theta^{\lambda}}[\cdot]$ denotes expectation under $\mathcal{P}_{\theta^{\lambda}}$.

- Write $\rho_i := \rho(u, s, \mathbf{e}_i)$, $i = 1, 2, \cdots, N$, and $\boldsymbol{\rho} := (\rho_1, \rho_2, \cdots, \rho_N)' \in \Re^N$.

- **Proposition 2.** For each $i = 1, 2, \cdots, N$, let $\Delta_i^R := \frac{\partial \rho_i}{\partial s}$ and $\lambda(\Delta_i^R) = \begin{cases} \lambda_i^+ & \text{if } \Delta_i^R > 0 \\ \lambda_i^- & \text{if } \Delta_i^R < 0 \end{cases}$. Then ρ_i, $i = 1, 2, \cdots, N$, satisfy the following system of N-coupled P.D.E.s:

$$\frac{\partial \rho_i}{\partial u} + \frac{1}{2}\sigma_i^2 s^2 \frac{\partial^2 \rho_i}{\partial s^2} + \lambda(\Delta_i^R) s \frac{\partial \rho_i}{\partial s} - r\rho_i + \langle \boldsymbol{\rho}, \mathbf{A}(t)\mathbf{e}_i \rangle = 0 ,$$

with the following terminal conditions:

$$\rho(t+h, S(t+h), \mathbf{e}_i) = e^{rh} V(t, S(t), \mathbf{X}(t)) - V(t+h, S(t+h), \mathbf{e}_i) .$$

- Adapt to evaluate the GSE for the residual risk due to incomplete delta-neutral hedging by one more state variable

- Apply to a barrier option (boundary conditions)

§4. The GSE for an American option

- Given $\mathbf{S}(t) = s$ and $\mathbf{X}(t) = \mathbf{x}$, the conditional price of an American option at time t with a finite maturity T is:

  $$V^a(t, s, \mathbf{x}) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathsf{E}^{\theta}[e^{-r(\tau - t)} g(\tau, S(\tau)) | S(t) = s, \mathbf{X}(t) = \mathbf{x}] \ ,$$

  where $g(t, S(t))$ is the intrinsic value of the American option at time t and θ is the risk-neutralized process in the regime-switching Esscher transform; $\mathcal{T}_{t,T}$ denotes the space of all G-stopping time taking values in $[t, T]$.

- **Lemma 3:** Let $V_i^a := V^a(t, s, \mathbf{e}_i)$, $i = 1, 2, \cdots, N$, and $\mathbf{V}^a := (V_1^a, V_2^a, \cdots, V_N^a)' \in \Re^N$. Then, V_i^a, $i = 1, 2, \cdots, N$, satisfy the following system of N-coupled variational inequalities:

$$\frac{\partial V_i^a}{\partial t} + \frac{1}{2}\sigma_i^2 s^2 \frac{\partial^2 V_i^a}{\partial s^2} + r\, s \frac{\partial V_i^a}{\partial s} - r\, V_i^a + \langle \mathbf{V}^a, \mathbf{A}(t)\mathbf{e}_i \rangle \leq 0 \ ,$$

$$V^a(t, s, \mathbf{e}_i) \geq g(t, s) \ ,$$

$$\left(\frac{\partial V_i^a}{\partial t} + \frac{1}{2}\sigma_i^2 s^2 \frac{\partial^2 V_i^a}{\partial s^2} + r\, s \frac{\partial V_i^a}{\partial s} - r\, V_i^a + \langle \mathbf{V}^a, \mathbf{A}(t)\mathbf{e}_i \rangle \right) (V_i^a - g) = 0 \ ,$$

and

$$V^a(T, s, \mathbf{e}_i) = g(T, s) \ .$$

- Variational inequalities for optimal stopping: Bensoussan and Lions (1982)

- Allow the possibility of early exercise at any time $u \in [t, t+h]$

- $\mathcal{T}_{u,t+h}$: the space of all G-stopping times taking value in $[u, t+h]$.

- Future net loss of the American option over $[t, \tau^* \wedge (t+h)]$

$$\begin{aligned} & \Delta V^a(t, \tau^* \wedge (t+h) - t) \\ := \; & e^{r(\tau^* \wedge (t+h) - t)} V^a(t, S(t), \mathbf{X}(t)) \\ & - V^a(\tau^* \wedge (t+h) - t, S(\tau^* \wedge (t+h) - t), \mathbf{X}(\tau^* \wedge (t+h) - t)) \; . \end{aligned}$$

- The GSE for the American option conditional on $S(u) = s$ and $\mathbf{X}(u) = \mathbf{x}$:

$$\begin{aligned} & \rho^a(u, s, \mathbf{x}) \\ := \; & \sup_{\lambda \in \Theta, \tau^* \in \mathcal{T}_{u,t+h}} \mathsf{E}^{\theta^\lambda}[e^{-r(\tau^* \wedge (t+h) - u)} \Delta V^a(t, \tau^* \wedge (t+h) - t) | S(u) = s, \mathbf{X}(u) = \mathbf{x}] \; . \end{aligned}$$

- **Proposition 4:** Let $\rho_i^a := \rho^a(u, s, \mathbf{e}_i)$, $i = 1, 2, \ldots, N$, and $\boldsymbol{\rho}^a := (\rho_1^a, \rho_2^a, \cdots, \rho_N^a)$. Then, ρ_i, $i = 1, 2, \cdots, N$, satisfy the following system of N-coupled variational inequalities:

$$\frac{\partial \rho_i^a}{\partial u} + \frac{1}{2}\sigma_i^2 s^2 \frac{\partial^2 \rho_i^a}{\partial s^2} + \lambda(\Delta_i^{R,a}) s \frac{\partial \rho_i^a}{\partial s} - r\,\rho_i^a + \langle \boldsymbol{\rho}^a, \mathbf{A}(t)\mathbf{e}_i \rangle \leq 0 \ ,$$

$$\rho^a(u, S(u), \mathbf{e}_i) \geq -g(u, S(u)) + e^{r(u-t)} V^a(t, S(t), \mathbf{X}(t)) := \tilde{g}(u, S(u)) \ ,$$

$$\rho^a(t+h, S(t+h), \mathbf{e}_i) = e^{rh} V^a(t, S(t), \mathbf{X}(t)) - V^a(t+h, S(t+h), \mathbf{e}_i) \ ,$$

$$\left(\frac{\partial \rho_i^a}{\partial u} + \frac{1}{2}\sigma_i^2 s^2 \frac{\partial^2 \rho_i^a}{\partial s^2} + \lambda(\Delta_i^{R,a}) s \frac{\partial \rho_i^a}{\partial s} - r\,\rho_i^a + \langle \boldsymbol{\rho}^a, \mathbf{A}(t)\mathbf{e}_i \rangle \right) (\rho_i^a - \tilde{g}) = 0 \ ,$$

where

$$\Delta_i^{R,a} = \frac{\partial \rho_i^a}{\partial s} \ , \quad \lambda(\Delta_i^{R,a}) = \begin{cases} \lambda_i^+ & \text{if} \ \ \Delta_i^{R,a} > 0 \\ \lambda_i^- & \text{if} \ \ \Delta_i^{R,a} < 0 \end{cases} \ .$$

§5. Some remarks

- One more dimension: Risk in addition to price and hedge ratio

- In addition to models, human factors are important; Lesson from LTCM

- Profit testing: Combine subjective scenarios with quantitative models

- Extend to the case of Markov-modulated pure jump price processes; infinite jump activities; relate to VG and CGMY processes; empirical justification by Taylor (2008)

§6. Summary

- Developed a two-stage procedure for evaluating the GSE for the unhedged risk of an option position under a Markovian regime-switching framework

- Used the regime-switching Esscher transform for specifying a price kernel for marking to the model and for generating a family of subjective probabilities for risk measurement

- Adapted the method to complex derivatives, such as American options

§7. Future Research

- Finance, Economics and Actuarial Science:

  1. Applications to Corporate Finance: Merton's firm value model; Executive stock options

  2. Energy derivatives, weather derivatives, credit derivatives

  3. Applications to equity-linked insurance products: Partially hedged positions and residual risks

  4. Deterministic models for economic cycles: Discrete non-linear dynamical systems or the skeleton of a Markov chain

- Technical Issues:

    1. Numerical methods for P.D.E.s and variational inequalities

    2. Viscosity solution and stability of numerical scheme

    3. Sensitivities of risk measures: Malliavin calculus

    4. Extend to the case of Hidden Markov chain; Filtering method and Bayesian MCMC approach

Thank you!

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