A Bivariate Shot Noise Self-Exciting Process for Insurance

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Overview

- A catastrophic event such as flood, storm, hail, bushfire, earthquake and terrorism brings about huge losses in properties, motor vehicles and from the interruption of businesses.

- Particular examples concern losses due to 2011 Great Eastern Japan Earthquake and Tsunami, 2010-2011 Queensland floods, 2009 Victorian Bushfires (Report of 2009 Victorian Bushfires Royal Commission, 2010), 2005 Hurricane Katrina (Burton and Hicks 2005) and 2001 September 11 attack (Makinen 2002).

- These are extreme risks which pose a new challenge to the financial viability of insurers.

Overview (continued)

- Due to global warming and climate changes, it is inevitable that more extreme losses will occur from catastrophic events. To accommodate the clustering of losses due to increases in frequency and intensity of natural/man-made disasters, improved models are required to predict losses arising from catastrophic events.

- For that purpose, *a bivariate shot noise Hawkes process* is introduced in which both *externally excited joint* jumps following *a homogeneous Poisson process* and two separate *self-excited* jumps which themselves follow *Poisson cluster processes*, are used.

Self-exciting (or Hawkes) processes

- Self-exciting or Hawkes processes (Hawkes 1971, Hawkes and Oakes 1974 and Daley and Vere-Jones 2003) are versatile point processes, interesting both from a theoretical as well as a practical point view.

- The theoretical foundation of Hawkes processes can be traced from a series of paper written by Brémaud and Massoulié (1996, 2001 and 2002) and Liniger (2009).

- Relevant publications in seismology and the modelling of the occurrence of earthquakes are Vere-Jones (1975), Adamopoulos (1976), Vere-Jones (1978), Ozaki (1979), Vere-Jones and Ozaki (1982), Ogata (1988).

Self-exciting (or Hawkes) processes (continued)

- The applications and modelling of Hawkes processes in *finance* can be found in Chavez-Demoulin et al. (2005), McNeil et al. (2005), Bauwens and Hautsch (2009), Bowsher (2007), Aït-Sahalia et al. (2010) and Embrechts et al. (2011).

- *Credit default* modelling using these processes can be noticed in Errais et al. (2010), Giesecke and Kim (2011) and Dassios and Zhao (2011).

- Stabile and Torrisi (2010) apply Hawkes process in an *insurance* risk context to study the asymptotic behavior of infinite and finite horizon ruin probabilities.

A Self-exciting process

$$H_t = H_0 + \sum_{i \geq 1} X_i \mathbb{I}\left(T_i \leq t\right) + \sum_{j \geq 1} Y_j \mathbb{I}\left(T_j \leq t\right),$$

where $H_0 > 0$ is the initial value of a self-excited process, $\mathbb{I}$ is the indicator function, $\{X_i\}_{i=1,2,\cdots}$ is a sequence of independent identical distributed positive **externally excited jumps** with distribution function $F(x)$, $x > 0$ at the corresponding random times $\{T_i\}_{i=1,2,\cdots}$ following a homogeneous Poisson process N_t with constant intensity $\rho > 0$ and $\left\{Y_j\right\}_{j=1,2,\cdots}$ is a sequence of independent identically distributed positive **self-excited jumps** with distribution function $G(y)$, $y > 0$, at the corresponding random times $\left\{T_j\right\}_{j=1,2,\cdots}$.

An univariate shot noise self-exciting process

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i \geq 1} X_i e^{-\delta(t-T_i)} \mathbb{I}\left(T_i \leq t\right) + \sum_{j \geq 1} Y_j e^{-\delta\left(t-T_j\right)} \mathbb{I}\left(T_j \leq t\right),$$

where $\lambda_0 > 0$ is a constant as the initial value of an univariate shot noise self-exciting process, $\delta > 0$ is a constant rate of exponential decay and all symbols have been previously defined.

Definition

From

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i \geq 1} X_i e^{-\delta(t - T_i)} \mathbb{I}\left(T_i \leq t\right) + \sum_{j \geq 1} Y_j e^{-\delta\left(t - T_j\right)} \mathbb{I}\left(T_j \leq t\right),$$

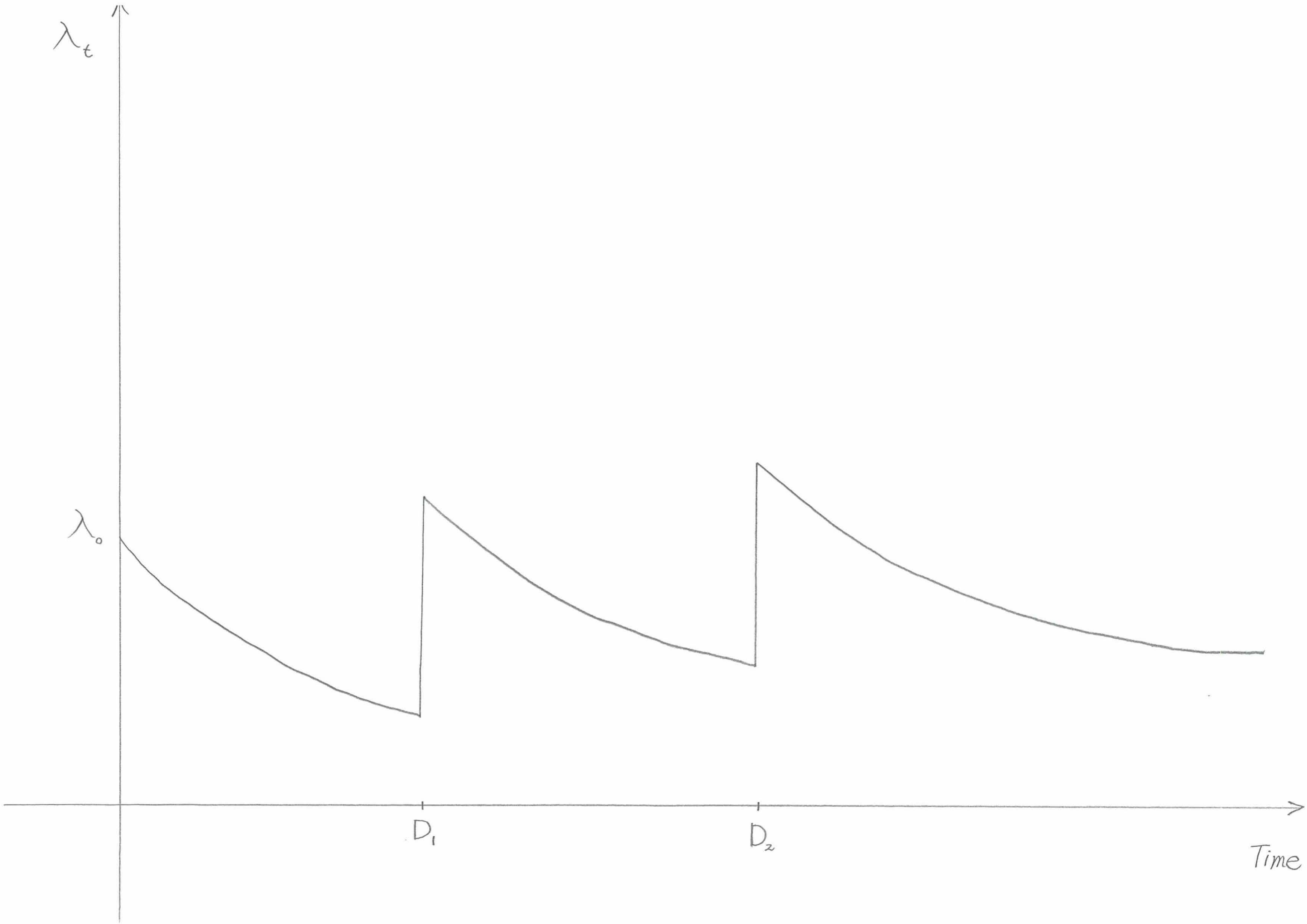
let us **firstly** look at

$$\lambda_0 e^{-\delta t} + \sum_{i \geq 1} X_i e^{-\delta(t - T_i)} \mathbb{I}\left(T_i \leq t\right).$$

Definition (continued): *Immigrants*

$$\lambda_0 e^{-\delta t} + \sum_{i \geq 1} X_i e^{-\delta(t-T_i)} \mathbb{I}(T_i \leq t),$$

where **externally excited jumps** $\{X_i\}_{i=1,2,\cdots}$ with distribution function $F(x)$, $x > 0$ at the corresponding random times $\{T_i\}_{i=1,2,\cdots}$ following a homogeneous **Poisson** process N_t with points $D_i \in (0, \infty)$ and constant intensity $\rho > 0$. Each jump/point/birth is called an *immigrant* and this is non-self exciting jump. These immigrants (i.e. jumps/births) form the points of **generation 0**.



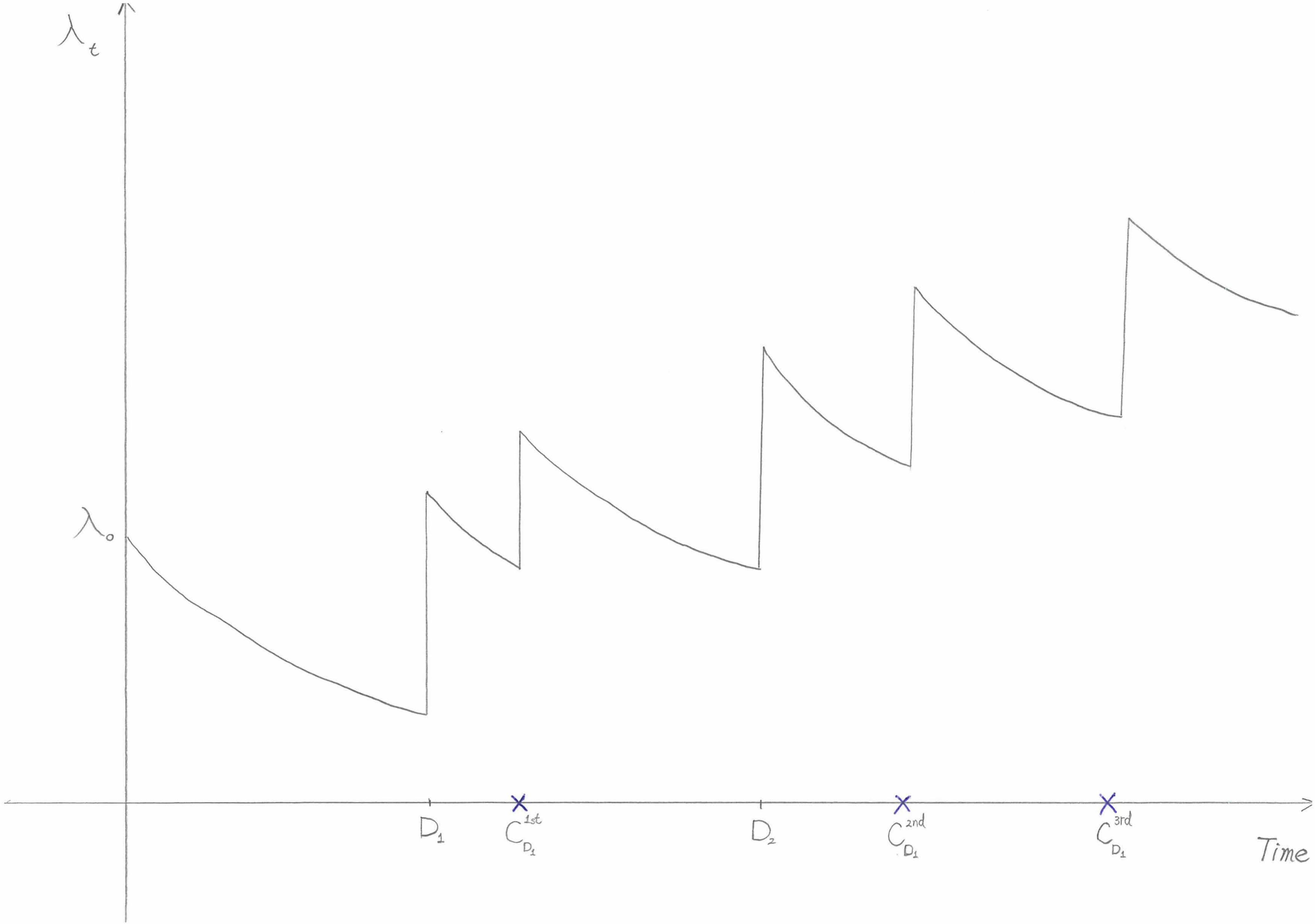
λ_t
λ_0
D_1
D_2
Time

Definition (continued): *Offsprings*

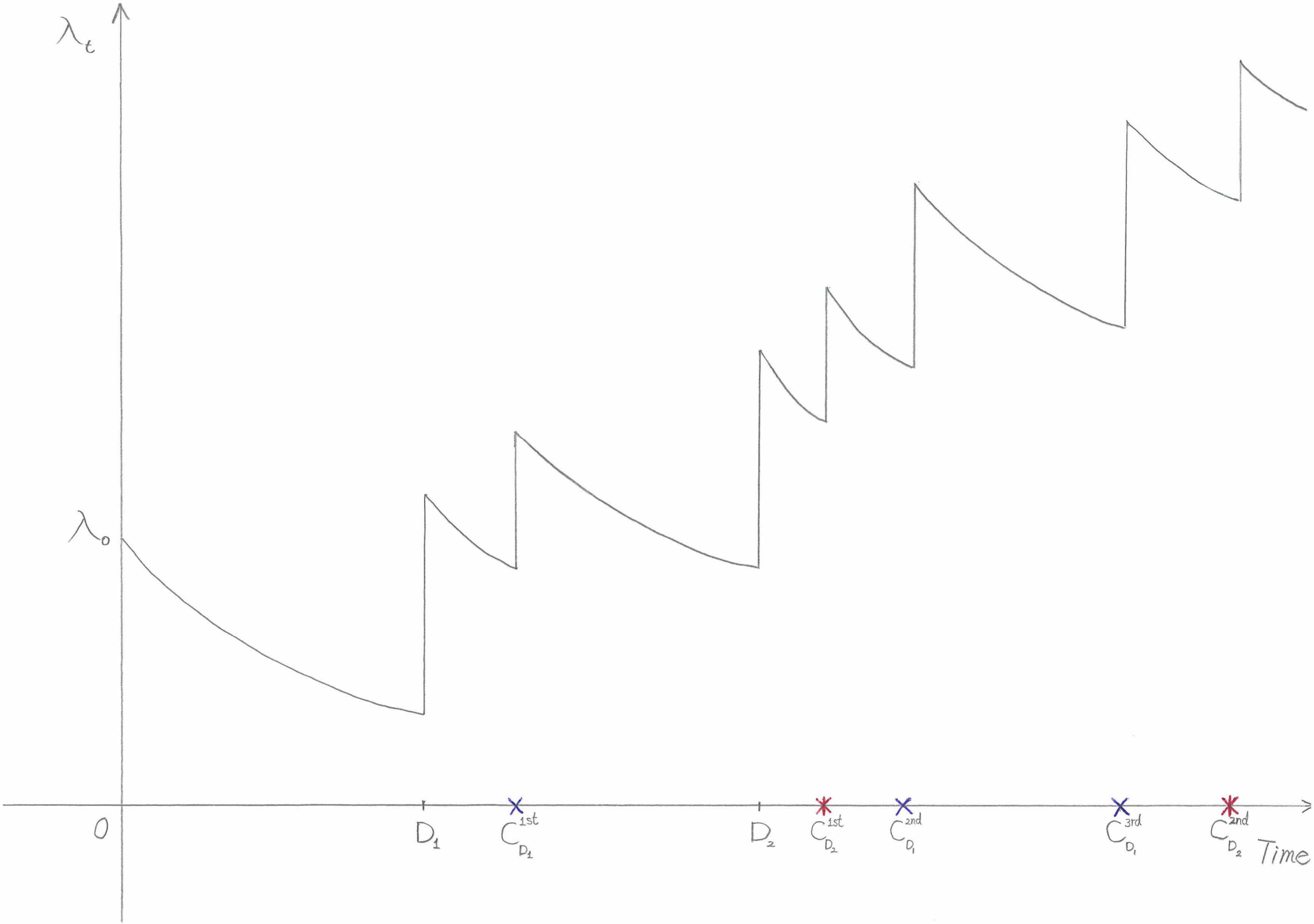
- Each immigrants D_i generates **a cluster** $C_i = C_{D_i}$ which the random set formed by the points of generations $0, 1, 2, \cdots$ with the following branching structure: The immigrant D_i is said to be of generation 0.

- Given generations $0, 1, 2, \cdots, r$ in C_i, each point $T_j \in C_i$ of **generation** r generates a Poisson process on $\left(T_j, \infty\right)$ of *offspring* of **generation** $r+1$ with intensity function $Y_j e^{-\delta\left(\cdot\, - T_j\right)}$, where a positive self-excited jump Y at time T_j has distribution function $G(y)$, $y > 0$, independent of the points of generation $0, 1, 2, \cdots, r$. Hence we have

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i \geq 1} X_i e^{-\delta(t-T_i)} \mathbb{I}\,(T_i \leq t) + \sum_{j \geq 1} Y_j e^{-\delta\left(t-T_j\right)} \mathbb{I}\left(T_j \leq t\right).$$



λ_t
λ_0
D_1
$C^{1st}_{D_1}$
D_2
$C^{2nd}_{D_1}$
$C^{3rd}_{D_1}$
Time



λ_t
λ_0
0
D_1
$C^{1st}_{D_1}$
D_2
$C^{1st}_{D_2}$
$C^{2nd}_{D_1}$
$C^{3rd}_{D_1}$
$C^{2nd}_{D_2}$
Time

Insurance application of univariate shot noise self-exciting process

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i \geq 1} X_i e^{-\delta(t-T_i)} \mathbb{I}(T_i \leq t) + \sum_{j \geq 1} Y_j e^{-\delta\left(t-T_j\right)} \mathbb{I}\left(T_j \leq t\right). \quad (*)$$

Replace $-\delta$ with δ in (*) denoting λ_t by L_t, which is the **accumulated value** of aggregate losses, then it is given by

$$L_t = L_0 e^{\delta t} + \sum_{i \geq 1} X_i e^{\delta(t-T_i)} \mathbb{I}(T_i \leq t) + \sum_{j \geq 1} Y_j e^{\delta\left(t-T_j\right)} \mathbb{I}\left(T_j \leq t\right), \quad (**)$$

where δ is now the *force of interest rate*.

Insurance application of univariate shot noise self-exciting process (continued)

Multiply by $e^{-\delta t}$ in (**) (i.e. $L_t^0 = L_t e^{-\delta t}$), then the **discounted value** of the aggregate losses with initial loss amount \$$L_0$ is given by

$$L_t^0 = L_0 + \sum_{i \geq 1} X_i e^{-\delta T_i} \mathbb{I}\left(T_i \leq t\right) + \sum_{j \geq 1} Y_j e^{-\delta T_j} \mathbb{I}\left(T_j \leq t\right).$$

We can interpret the process as follows: the standard losses, $\{X_i\}_{i=1,2,\cdots}$ that occur according to a homogeneous Poisson process N_t with constant intensity $\rho > 0$ trigger after-losses (after-shock losses), $\left\{Y_j\right\}_{j=1,2,\cdots}$ according to the branching structure described previously, which are unknown at the arrival times of standard losses from a catastrophic event in practice. The force of interest rate δ is used to discount all losses.

Insurance premium

Based on the piecewise deterministic Markov process theory developed by Davis (1984), and the martingale methodology used by Dassios and Jang (2003), the expectation of the **discounted value** of the aggregate losses is given by

$$E\left(L_t^0 \mid L_0\right) = \left[L_0 + \mu_{1_F}\rho \left\{\frac{1-e^{-\left(\delta-\mu_{1_G}\right)t}}{\delta-\mu_{1_G}}\right\}\right] e^{\mu_{1_G}t} = \left(L_0 + \mu_{1_F}\rho\ \bar{a}_{\overline{t|}\ \text{at}\ \delta+\mu_{1_G}}\right) e^{\mu_{1_G}t},$$

which can be considered as the net *insurance premium* at time $t = 0$ including the effect of the interest rate, where

$$\mu_{1_F} = \int_0^\infty x dF(x) \ \text{ and } \ \mu_{1_G} = \int_0^\infty y dG(y).$$

Numerical Example 1

We assume that a Property Insurance Company's (or a State Government's) standard loss frequency rate is 50 per unit time period (say, per year) with the average of losses 1. The mean of after-losses (after-shock losses), which are unknown at the arrival times of standard losses from a catastrophic event, (e.g. an earthquake) is assumed to be 2. We assume that the force of interest rate is 0.05 and that an initial loss amount that has been carried over is 1. Using an exponential distribution for $F(x)$ and $G(y)$, respectively, i.e. $F(x) = 1 - e^{-\alpha x}$ and $G(y) = 1 - e^{-\gamma y}$ with $\alpha > 0$, $\gamma > 0$, then the parameter values to calculate the expectation of the **discounted value** of the aggregate losses are $L_0 = 1$, $\delta = 0.05$, $\rho = 50$, $\alpha = 1$, $\gamma = 0.5$, $t = 1$.

Numerical Example 1 (continued)

Table 1

$\left(L_0 + \mu_{1_F}\rho\ \bar{a}_{\overline{t\rvert}\ \text{at}\ \delta^{(1)}+\mu_{1_G}}\right) e^{\mu_{1_G} t}$	$L_0+\mu_{1_F}\rho\bar{a}_{\overline{t\rvert}\ \text{at}\ \delta^{(1)}}$	$L_0+\mu_{1_F}\rho t$	$L_0 e^{\mu_{1_G} t}$
\$164.41	\$49.771	\$51	\$7.3891

Table 1 shows that the expected discounted premium value 164.41 calculated based on shot noise self-exciting process is more than **three-times** higher than its counterpart 49.771 calculated based on compound Poisson process. It is because the expected discounted premium of grows exponentially. '$L_0\ e^{\mu_{1_G} t}$' is the expected discounted premium after eliminating the frequency rate for standard losses, which also grows **exponentially**.

Numerical Example 1 (continued)

Table 1

$\left(L_0 + \mu_{1_F}\rho\ \bar{a}_{\overline{t}\rvert \text{ at } \delta^{(1)}+\mu_{1_G}}\right) e^{\mu_{1_G} t}$	$L_0+\mu_{1_F}\rho\bar{a}_{\overline{t}\rvert \text{ at } \delta^{(1)}}$	$L_0+\mu_{1_F}\rho t$	$L_0 e^{\mu_{1_G} t}$
\$164.41	\$49.771	\$51	\$7.3891

Given time t, μ_{1_G} (or equivalently $e^{\mu_{1_G} t}$) which is the mean of after-losses (after-shock losses), is the main driver to raise the expected discounted premium higher than its counterpart. Hence the significance of loss clustering impacts from a catastrophic event depends on after-loss (after-shock loss) size measure $dG(y)$. It will be of interest to examine the expected discounted premium value using other after-loss (after-shock loss) size measures.

Numerical Example 1 (continued)

Table 1

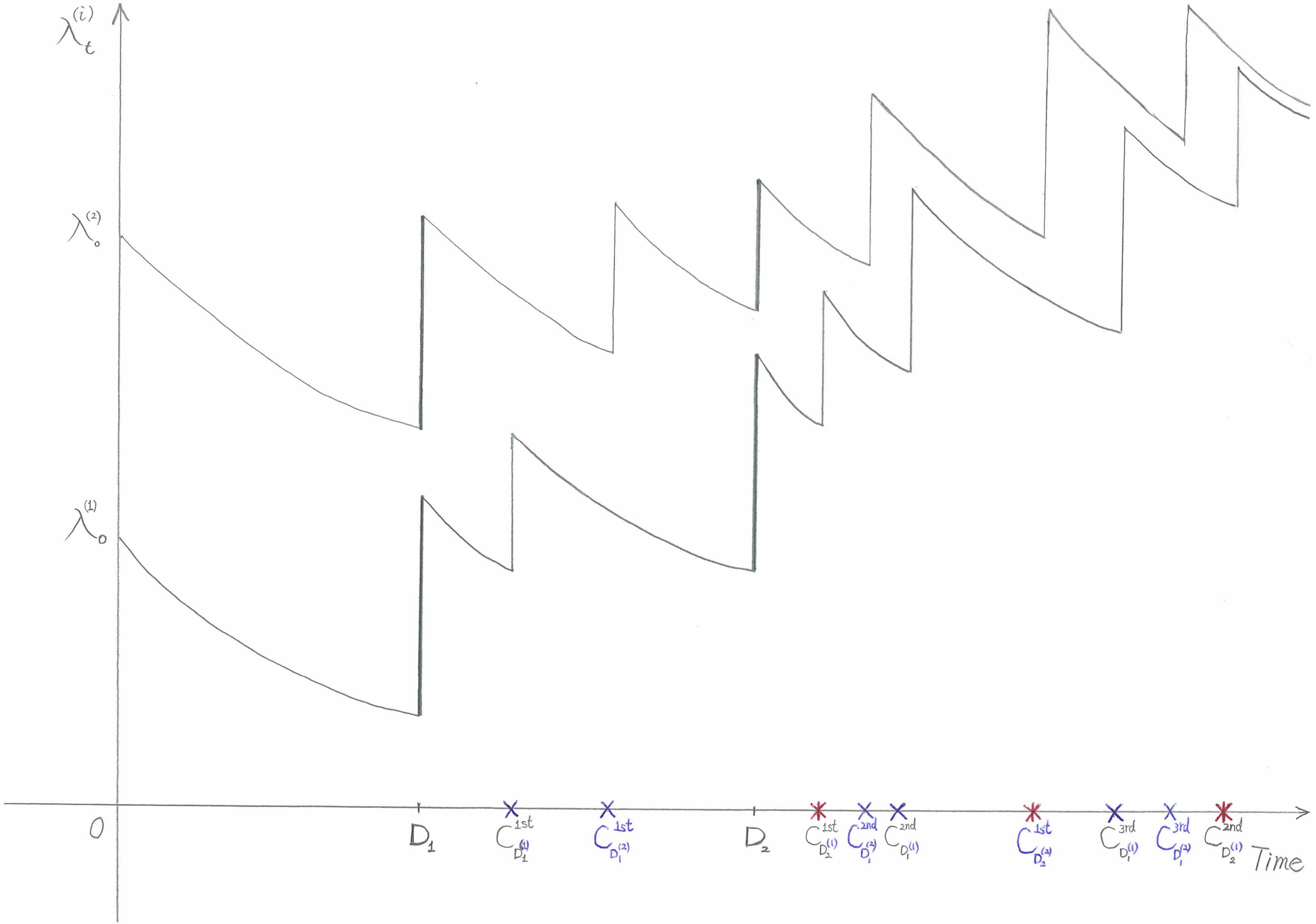
$\left(L_0 + \mu_{1_F}\rho \ \bar{a}_{\overline{t\vert} \text{ at } \delta^{(1)}+\mu_{1_G}}\right) e^{\mu_{1_G} t}$	$L_0+\mu_{1_F}\rho\bar{a}_{\overline{t\vert} \text{ at } \delta^{(1)}}$	$L_0+\mu_{1_F}\rho t$	$L_0 e^{\mu_{1_G} t}$
\$164.41	\$49.771	\$51	\$7.3891

The expected discounted premium value calculated based on shot noise self-exciting process clearly justifies that it can be used modelling discounted aggregate losses from **catastrophic events** to accommodate loss clustering due to increases in frequency and intensity of natural and man-made disasters in practice.

A bivariate shot noise self-exciting process

$$\lambda_t^{(1)} = \lambda_0^{(1)} e^{-\delta^{(1)} t} + \sum_{i \geq 1} X_i^{(1)} e^{-\delta^{(1)}\left(t - T_{1,i}\right)} I\left(T_{1,i} \leq t\right) + \sum_{j \geq 1} Y_j e^{-\delta^{(1)}\left(t - T_{2,j}\right)} I\left(T_{2,j} \leq t\right),$$

$$\lambda_t^{(2)} = \lambda_0^{(2)} e^{-\delta^{(2)} t} + \sum_{i \geq 1} X_i^{(2)} e^{-\delta^{(2)}\left(t - T_{1,i}\right)} I\left(T_{1,i} \leq t\right) + \sum_{k \geq 1} Z_k e^{-\delta^{(2)}\left(t - T_{2,k}\right)} I\left(T_{2,k} \leq t\right)$$



$\lambda_t^{(i)}$
$\lambda_0^{(2)}$
$\lambda_0^{(1)}$
0
D_1
$C^{1st}_{D_1^{(1)}}$
$C^{1st}_{D_1^{(2)}}$
D_2
$C^{1st}_{D_2^{(1)}}$
$C^{2nd}_{D_1^{(2)}}$
$C^{2nd}_{D_1^{(1)}}$
$C^{1st}_{D_2^{(2)}}$
$C^{3rd}_{D_1^{(1)}}$
$C^{3rd}_{D_1^{(2)}}$
$C^{2nd}_{D_2^{(1)}}$
Time

Insurance application of **bivariate** shot noise self-exciting process

$$L_t^{0(1)} = L_0^{(1)} + \sum_{i\geq 1} X_i^{(1)} e^{-\delta T_i} I\left(T_i \leq t\right) + \sum_{j\geq 1} Y_j e^{-\delta T_j} I\left(T_j \leq t\right),$$

$$L_t^{0(2)} = L_0^{(2)} + \sum_{i\geq 1} X_i^{(2)} e^{-\delta T_i} I\left(T_i \leq t\right) + \sum_{k\geq 1} Z_k e^{-\delta T_k} I\left(T_k \leq t\right).$$

An economic interpretation from the perspective of the cluster process representation for bivariate shot noise self-exciting process is the following: As a result of a catastrophic event, such as flood, storm, hail, bushfire and earthquake, joint losses of properties and motors (or businesses interruption) $\begin{bmatrix} X_i^{(1)} \\ X_i^{(2)} \end{bmatrix}_{i=1,2,\cdots}$ occur **simultaneously/collaterally** according to a homogeneous Poisson process

N_t with constant intensity $\rho > 0$. In the aftermath of each joint losses to this company, they could further trigger a series of after-losses (after-shock losses), $\left\{Y_j\right\}_{j=1,2,\cdots}$ and $\{Z_k\}_{k=1,2,\cdots}$ according to the branching structure described previously, which are unknown at the arrival times of each joint losses. The force of interest rate δ is used to discount all losses. We have

$$E\left(L_t^{0\,(1)} \mid L_0^{(1)}\right) = \left(L_0^{(1)} + \mu_{1_{F_1}}\rho\ \bar{a}_{\overline{t|}\text{ at }\delta+\mu_{1_G}}\right) \times e^{\mu_{1_G}t},$$

$$E\left(L_t^{0\,(2)} \mid L_0^{(2)}\right) = \left(L_0^{(2)} + \mu_{1_{F_2}}\rho\ \bar{a}_{\overline{t|}\text{ at }\delta+\mu_{1_H}}\right) \times e^{\mu_{1_H}t}.$$

Insurance premium

Let us assume that an insurance company charges collateral loss insurance premium as follows:

$$
\begin{aligned}
& E\left(L_t^{0\,(1)} + L_t^{0\,(2)} \mid L_0^{(1)}, L_0^{(2)}\right) + \phi\sqrt{Var\left(L_t^{0\,(1)} + L_t^{0\,(2)} \mid L_0^{(1)}, L_0^{(2)}\right)} \\
= \; & E\left(L_t^{0\,(1)} \mid L_0^{(1)}\right) + E\left(L_t^{0\,(2)} \mid L_0^{(2)}\right) \\
& +\phi\sqrt{Var\left(L_t^{0\,(1)} \mid L_0^{(1)}\right) + Var\left(L_t^{0\,(2)} \mid L_0^{(2)}\right) + 2Cov\left(L_t^{0\,(1)}, L_t^{0\,(2)} \mid L_0^{(1)}, L_0^{(2)}\right)},
\end{aligned}
$$

where $0 \leq \phi \leq 1$ and $\phi\sqrt{Var\left(L_t^{0\,(1)} + L_t^{0\,(2)} \mid L_0^{(1)}, L_0^{(2)}\right)}$ can be considered as a security loading.

Covariance: $Cov\left(L_t^{0\ (1)}, L_t^{0\ (2)} \mid L_0^{(1)}, L_0^{(2)}\right)$

$$Cov\left(L_t^{0\ (1)}, L_t^{0\ (2)} \mid L_0^{(1)}, L_0^{(2)}\right) =$$
$$E\left(L_t^{0\ (1)}\ L_t^{0\ (2)} \mid L_0^{(1)}, L_0^{(2)}\right) - E\left(L_t^{0\ (1)} \mid L_0^{(1)}\right) E\left(L_t^{0\ (2)} \mid L_0^{(2)}\right)$$

To calculate the covariance easier, we use the Farlie-Gumbel-Morgenstern (FGM) copulas given by $C(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2)$, where $u_1 \in [0, 1]$, $u_2 \in [0, 1]$ and $\theta \in [-1, 1]$, with $F\left(x^{(1)}\right) = 1 - e^{-\alpha x^{(1)}}$ $(\alpha > 0)$ and $F\left(x^{(2)}\right) = 1 - e^{-\beta x^{(2)}}$ $(\beta > 0)$, from which we have $E\left(X^{(1)} X^{(2)}\right) = \int_0^\infty \int_0^\infty x^{(1)} x^{(2)} dF\left(x^{(1)}, x^{(2)}\right) = \frac{1}{\alpha\beta}\left(1 + \frac{\theta}{4}\right)$. We also use an exponential distribution for $H(z)$, i.e. $H(z) = 1 - e^{-\zeta z}$ with $\zeta > 0$.

Numerical Example 2

- We assume that a Property and Motor (or Businesses Interruption) insurance company's (or a State Government's) standard loss frequency rate is 50 per unit time period (say, per year) with two average of losses 1 and 2, respectively.

- The mean of after-losses (after-shock losses) from Motor (or Businesses Interruption) insurance side, which are unknown at the arrival times of joint standard losses from a catastrophic event (e.g. an earthquake), is assumed to be 4.

- We assume that an initial loss amount that has been carried over from Motor (or Businesses Interruption) insurance side is 1.

- As the security loading factor, this insurance company uses 0.5, i.e. $\phi = 0.5$ and the force of interest rate, $\delta = \delta^{(1)} = \delta^{(2)} = 0.05$.

- Hence the parameter values used for Motor (or Businesses Interruption) insurance side are $L_0^{(2)} = 1$, $\beta = 0.5$, $\zeta = 0.25$.

- Using the parameter values in Example 1 for Property insurance side, i.e. $L_0^{(1)} = 1$, $\rho = 50$, $\alpha = 1$, $\gamma = 0.5$, $t = 1$, collateral loss insurance premium calculations are shown in Table 2.

Table 2	Collateral loss insurance premium	
θ	Bivariate shot noise self-exciting (discounted)	Bivariate shot noise Poisson (discounted)
1	\$1858.8	\$159.75
0.5	\$1858.2	\$159.49
0	\$1857.5	\$159.22
-0.5	\$1856.9	\$158.94
-1	\$1856.2	\$158.66

Table 2 shows that collateral loss insurance premium values calculated using bivariate shot noise self-exciting process (discounted) are **significantly higher** than their counterparts calculated using bivariate shot noise Poisson process (discounted) at different value of θ. It is because two "exponential growths", i.e. $e^{\mu_{1_G} t}$ and $e^{\mu_{1_H} t}$.

Linear correlation coefficient

Table 3	$Corr\left(L_t^{0\,(1)}, L_t^{0\,(2)} \mid L_0^{(1)}, L_0^{(2)}\right)$	
θ	Bivariate shot noise self-exciting (discounted)	Bivariate shot noise Poisson (discounted)
1	0.22791	0.88362
0.5	0.20525	0.79520
0	0.18231	0.70675
-0.5	0.15964	0.61833
-1	0.13670	0.52991

Table 3 shows that the linearities between $L_t^{0\,(1)}$ and $L_t^{0\,(2)}$ calculated using bivariate shot noise self-exciting process (discounted) are significantly lower than their counterparts calculated using bivariate shot noise Poisson process (discounted) at different value of θ. It is because two separate loss clustering impacts (i.e. two separate after-shock losses' impacts) weaken the linearity between ${L_t^0}^{(1)}$ and ${L_t^0}^{(2)}$. Therefore it will be also of interest to compare bivariate distribution for shot noise self-exciting (discounted) case with its counterpart, in particular seeing their two tail corners inverting bivariate Fast Fourier transforms using bivariate Laplace transforms.

The generator of the process $\left(L^{(1)}, L^{(2)}, t\right)$

$$A\,f\left(L^{(1)}, L^{(2)}, t\right) = \frac{\partial f}{\partial t} + \delta^{(1)} L^{(1)} \frac{\partial f}{\partial L^{(1)}} + \delta^{(2)} L^{(2)} \frac{\partial f}{\partial L^{(2)}}$$

$$+\rho\left[\int_0^\infty \int_0^\infty f\left(L^{(1)}+x^{(1)}, L^{(2)}+x^{(2)}, t\right) dF\left(x^{(1)}, x^{(2)}\right) - f\left(L^{(1)}, L^{(2)}, t\right)\right]$$

$$+L^{(1)}\left[\int_0^\infty f\left(L^{(1)}+y, L^{(2)}, t\right) dG(y) - f\left(L^{(1)}, L^{(2)}, t\right)\right]$$

$$+L^{(2)}\left[\int_0^\infty f\left(L^{(1)}, L^{(2)}+z, t\right) dH(z) - f\left(L^{(1)}, L^{(2)}, t\right)\right].$$

The significance of two separate loss clustering impacts (i.e. two separate after-shock losses' impacts) from a catastrophic event depends on two after-shock loss size distributions $dG(y)$ and $dH(z)$ as well as standard joint loss size distribution $dF\left(x^{(1)}, x^{(2)}\right)$ and its frequency rate ρ. It will be of interest to examine collateral loss insurance premium values using other after-shock loss size distributions and other standard joint loss size distributions.

Future research

- Spillover effects and modelling systemic risk.
- Modelling security market events.
- Modelling operational risk with the extension of dimension.
- Adding diffusion components to price financial derivatives.
- Replacing a Poisson process with a Cox process.