Risk Measures for Derivative Securities: From a Yin-Yang Approach to Aerospace Space

Tak Kuen Siu *

*Department of Applied Finance and Actuarial Studies, Faculty of Business and Economics, Macquarie University, Sydney, AUSTRALIA
A Brief History of Binomial Tree

- Yin-Yang: I-Ching or Zhouyi (1,000 BC or before) and Tao-ism (the late 4th century BC)

- Origin in Probability Theory: Daniel Bernoulli (29 January 1700 - 17 March 1782); Coin tossing experiment \{H, T\}

- Discrete-time binomial tree in finance: Bill Sharpe?

- A beautiful paper by Cox, Ross and Rubinstein, CRR, (1979): Option valuation in a discrete-time binomial model
Behind the scene: Boyle, Siu and Yang (2002)

- Asian Financial Crisis in 1997: LTCM and derivative securities

- Reappraisal of Value at Risk (VaR): Non-Subadditivity

- Coherent risk measures by Artzner, Delbean, Eber and Heath (1999)

- Tail risk, expected shortfall and a research report in Bank of Japan by Yamai and Yoshiba (2002)
The Challenge

- Traditional theories in finance: Linear risk

- Capital Asset Pricing Model and Arbitrage Pricing Theory

- Bigger universe of nonlinear risk: not well-explored!

- Examples: Derivative securities and hedged funds

- Current Practice: Traders use Greek Letters, such as Delta, Gamma, Rho, ..., etc.

- Consider a discrete-time financial model consisting of a risk-free bond \( B \) and a stock \( S \)

- Deal with a European call option \( C \) written on \( S \) with strike price \( K \) and maturity \( T \)

- Build the two-level binomial model from the CRR binomial model

- Evaluate a coherent risk measure, namely Expected Shortfall (ES), for derivative securities
The Model

• Suppose \{0, 1, 2, \ldots, T\} is the time parameter set in the first level.

• For each time point \(k\) in the first level, \([k, k + 1]\) is the time interval for risk measurement.

• Divide \([k, k + 1]\) into \(m\) equal sub-intervals.

• Then \{0, 1, 2, \ldots, km, km + 1, \ldots, Tm\} is the time parameter set in the second level.
- For each sub-interval \([n, n + 1]\) in the second level, assume that, under a real-world probability measure \(\mathcal{P}\),

\[
\frac{B_{n+1}}{B_n} = \hat{r} \quad \text{and} \quad \frac{S_{n+1}}{S_n} = \begin{cases} 
  u & \text{with probability } p \\
  d & \text{with probability } 1 - p
\end{cases}
\]

- Call price from the CRR binomial model:

\[
C_{km} = \frac{1}{\hat{r}^{T_m - km}} \sum_{j=0}^{T_m - km} \binom{T_m - km}{j} q^j (1 - q)^{T_m - km - j} (S_{km} u^j d^{T_m - km - j} - K)^+
\]
**Expected Shortfall (ES) for the Call**

- $\Delta C_{k,m}$: the discounted net loss $C_{km} - \hat{r}^{-m}C_{(k+1)m}$ of the call option $C$ over $[km, (k + 1)m]$

- $\mathcal{F}_{km}$: the information generated by the values of $S$ up to and including time $km$

- **Under $\mathcal{P}$, the distribution of $\Delta C_{k,m}|\mathcal{F}_{km}$:**

  $$\Delta C_{k,m} = C_{km} - \hat{r}^{-m}C_{(k+1)m}(S_{km} u^j d^{m-j})$$

  with probability $\left(\begin{array}{c} m \\ j \end{array}\right) p^j (1 - p)^{m-j}$, $j = 0, 1, \ldots, m$. 
• ES for the call $C$:

$$
ES_\alpha(\Delta C_{k,m}|F_{km}) = E_P(\Delta C_{k,m}|I_{\{\Delta C_{k,m} \geq \text{VaR}_\alpha\}}, F_{km}) = \alpha^{-1}[E_P(\Delta C_{k,m}I_{\{\Delta C_{k,m} \geq \text{VaR}_\alpha}\}, F_{km}) + \text{VaR}_{\alpha,P}(\Delta C_{k,m}|F_{km})(\alpha - P(\Delta C_{k,m} \geq \text{VaR}_{\alpha,P}(\Delta C_{k,m}|F_{km})|F_{km})]
$$

• Adjustment for the discrete loss distribution to ensure the coherent property for the ES
An Expression for the ES

• Define \( j_\alpha = \sup\{j \in J \mid \Delta C_{k,m}(j) \geq \text{VaR}_{\alpha,P}(\Delta C_{k,m}|\mathcal{F}_{km})\} \), where \( J \) represents the set \( \{0, 1, 2, \ldots, m\} \). Then

\[
ES_{\alpha}(\Delta C_{k,m}|\mathcal{F}_{km}) = -\frac{1}{\hat{r}Tm-km \alpha} \left\{ \sum_{j=0}^{j_\alpha} \binom{m}{j} p^j (1 - p)^{m-j} \left[ \sum_{i=0}^{Tm-(k+1)m} \binom{Tm-(k+1)m}{i} (Tm-(k+1)m) \right] \right.

q^i(1 - q)^{Tm-(k+1)m-i} \left( S_{km}u^i d^{Tm-km-j-i} - K \right)^+ - \sum_{i=0}^{Tm-km} \binom{Tm-km}{i} (Tm-km)^+

\left. q^i(1 - q)^{Tm-km-i} \left( S_{km}u^i d^{Tm-km-i} - K \right)^+ \right\}

+ \Delta C_{k,m}(u^{j_\alpha} d^{m-j_\alpha}) \left[ 1 - \alpha^{-1} \sum_{j=0}^{j_\alpha} \binom{m}{j} p^j (1 - p)^{m-j} \right].
\]
Numerical Example

- Consider a European call with $T = 2$ months and $K = 22$. Suppose $S_0 = 25$

- Assume that the time horizon for measuring the risk of the position is one month

- $r = 0.7\%$ per month and $\sigma = 6\%$ per month

- Two-level binomial model: $m = 5$, $u = e^{0.0268}$, $d = e^{-0.0268}$ and $q = 0.5194$
The numerical values of ES and VaR for the call

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<th>$p$ \ $\alpha$</th>
<th>0.01</th>
<th>0.05</th>
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<td>0.7</td>
<td>2.185262 (1.989324)</td>
<td>1.591582 (0.852669)</td>
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Yang-Yin Grows Everything

- Yang-Yin generates many patterns

- Think about the modern computing technologies

- Central Limit Theorem: Binomial $\Rightarrow$ Normal

- CRR binomial model $\Rightarrow$ Continuous-time Black-Scholes-Merton model
Risk Measures for Derivatives in Continuous-Time Markets


• “Bang-Bang” type control: Use in Aerospace engineering

• Paul Wilmott’s book on Quantitative Finance and uncertain volatility models widely used in the finance industry
Risk Measures in Elliott, Siu and Chan (2008)

• Consider a financial model consisting of a bank account $B$ and a share $S$

• A continuous-time, $N$-state observable Markov chain $\{X(t)\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\{e_1, e_2, \ldots, e_N\}$.

• The price dynamics for $B$ and $S$ under $\mathcal{P}$:

$$
\begin{align*}
\text{d}B(t) &= rB(t)\text{d}t, \\
\text{d}S(t) &= \mu(t)S(t)\text{d}t + \sigma(t)S(t)\text{d}W(t),
\end{align*}
$$

where $\mu(t) := \langle \mu, X(t) \rangle$ and $\sigma(t) := \langle \sigma, X(t) \rangle$; $\mu := (\mu_1, \mu_2, \ldots, \mu_N)'$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)'$. 

First Step: Valuation


- The regime-switching Esscher transform by Elliott, Chan and Siu (2005):

  1. Define a process \( \theta := \{ \theta(t) \} \) by: \( \theta(t) = \langle \theta, X(t) \rangle \), where \( \theta = (\theta_1, \theta_2, \ldots, \theta_N)' \).

  2. The regime-switching Esscher transform \( Q_\theta \sim P \) associated with \( \theta := \{ \theta(t) \} \):

\[
\frac{dQ_\theta}{dP} \bigg|_{\mathcal{G}(t)} := \frac{\exp(\int_0^t \theta(u)dW(u))}{\mathbb{E}[\exp(\int_0^t \theta(u)dW(u))|F^X(t)]}.
\]
• Consider a European-style option with payoff \( V(S(T)) \) at maturity \( T \)

• Given \( S(t) = s \) and \( X(t) = x \), a conditional price of the option is given by:

\[
V(t, s, x) = \mathbb{E}^{\theta}[e^{-r(T-t)}V(S(T)) | S(t) = s, X(t) = x].
\]

• **Proposition 1:** Let \( V_i := V(t, s, e_i) \), for each \( i = 1, 2, \cdots, N \), and write \( V := (V_1, V_2, \cdots, V_N)' \in \mathbb{R}^N \). Write \( A(t) \) for the rate matrix of the chain at time \( t \). Then, \( V_i, i = 1, 2, \cdots, N \), satisfy the following system of \( N \)-coupled P.D.E.s:

\[
-rV_i + \frac{\partial V_i}{\partial t} + rs \frac{\partial V_i}{\partial s} + \frac{1}{2} \sigma_i^2 s \frac{\partial^2 V_i}{\partial s^2} + \langle V, A(t)e_i \rangle = 0,
\]

with terminal conditions \( V(T, s, e_i) = V(S(T)), i = 1, 2, \cdots, N \).
Second Step: Risk Evaluation

• For each $i = 1, 2, \cdots, N$, let $\Lambda_i = [\lambda_i^-, \lambda_i^+]$. For example, when $N = 2$ (i.e. State 1 is “Good Economy” and State 2 is “Bad Economy”), $\lambda_1^- = 0.05; \lambda_1^+ = 0.10; \lambda_2^- = 0.01; \lambda_2^+ = 0.05$.

• Suppose $\lambda(t)$ is the subjective appreciation rate of the share at time $t$. The chain modulates $\lambda(t)$ as:

$$\lambda(t) = \langle \lambda, X(t) \rangle,$$

where $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_N)' \in \mathbb{R}^N$ with $\lambda_i \in \Lambda_i, i = 1, 2, \cdots, N$. 
• Consider, for each $\lambda \in \Theta$, a process $\{\theta^\lambda(t)\}$ defined by putting
\[
\theta^\lambda(t) = \sum_{i=1}^{N} \left( \frac{\mu_i - \lambda_i}{\sigma_i} \right) \langle X(t), e_i \rangle.
\]

• The regime-switching Esscher transform $\mathcal{P}_{\theta^\lambda} \sim \mathcal{P}$ on $\mathcal{G}(t)$ with respect to $\{\theta^\lambda(t)\}$:
\[
\left. \frac{d\mathcal{P}_{\theta^\lambda}}{d\mathcal{P}} \right|_{\mathcal{G}(t)} := \frac{\exp(\int_0^t \theta^\lambda(u)dW(u))}{\mathbb{E}[\exp(\int_0^t \theta^\lambda(u)dW(u))|F^{X}(t)]}.
\]

• Under $\mathcal{P}_{\theta^\lambda}$,
\[dS(t) = \lambda(t)S(t)dt + \sigma(t)S(t)dW^\lambda(t),\]
where $\{W^\lambda(t)\}$ is a $(\mathcal{G}, \mathcal{P}_{\theta^\lambda})$-standard Brownian motion.
• Future net loss of the option position over \([t, t + h]\):

\[
\Delta V(t, h) := e^{rh}V(t, S(t), X(t)) - V(t + h, S(t + h), X(t + h))
\]

• Given \(S(u) = s\) and \(X(u) = x\), \(u \in [t, t + h]\), the generalized scenario expectation for the option position \(V\) over \([t, t + h]\):

\[
\rho(u, s, x) := \sup_{\lambda \in \Theta} E^{\theta^\lambda} \left[ \exp(-r(t + h - u))\Delta V(t, h) | S(u) = s, X(u) = x \right],
\]

where \(E^{\theta^\lambda}[\cdot]\) is an expectation under \(\mathcal{P}_{\theta^\lambda}\).

• Write \(\rho_i := \rho(u, s, e_i), i = 1, 2, \cdots, N\), and \(\rho := (\rho_1, \rho_2, \cdots, \rho_N)'\).
• **Proposition 2.** For each \( i = 1, 2, \ldots, N \), let \( \Delta_i^R := \frac{\partial \rho_i}{\partial s} \) and \( \lambda(\Delta_i^R) = \begin{cases} \lambda_i^+ & \text{if } \Delta_i^R > 0 \\ \lambda_i^- & \text{if } \Delta_i^R < 0 \end{cases} \). Then \( \rho_i, i = 1, 2, \ldots, N \), satisfy the following system of \( N \)-coupled P.D.E.s:

\[
\frac{\partial \rho_i}{\partial u} + \frac{1}{2} \sigma_i^2 s^2 \frac{\partial^2 \rho_i}{\partial s^2} + \lambda(\Delta_i^R) s \frac{\partial \rho_i}{\partial s} - r \rho_i + \langle \rho, A(t)e_i \rangle = 0,
\]

with the following terminal conditions:

\[
\rho(t+h, S(t+h), e_i) = e^{rh} V(t, S(t), X(t)) - V(t+h, S(t+h), e_i).
\]

• For the case of an American-style option, a system of coupled variational inequalities for the risk measures was obtained.
What Next?

- Incorporate credit risk and counterparty risk in the OTC markets

- Liquidity risk due to large trading positions

- Applications to modern insurance products with embedded options

References


