Risk Measures for Derivative Securities: From a Yin-Yang Approach to Aerospace Space

Tak Kuen Siu *

*Department of Applied Finance and Actuarial Studies, Faculty of Business and Economics, Macquarie University, Sydney, AUSTRALIA

A Brief History of Binomial Tree

- Yin-Yang: I-Ching or Zhouyi (1,000 BC or before) and Taoism (the late 4th century BC)

- Origin in Probability Theory: Daniel Bernoulli (29 January 1700 - 17 March 1782); Coin tossing experiment $\{H, T\}$

- Discrete-time binomial tree in finance: Bill Sharpe?

- A beautiful paper by Cox, Ross and Rubinstein, CRR, (1979): Option valuation in a discrete-time binomial model

Behind the scene: Boyle, Siu and Yang (2002)

- Asian Financial Crisis in 1997: LTCM and derivative securities

- Reappraisal of Value at Risk (VaR): Non-Subadditivity

- Coherent risk measures by Artnzer, Delbean, Eber and Heath (1999)

- Tail risk, expected shortfall and a research report in Bank of Japan by Yamai and Yoshiba (2002)

The Challenge

- Traditional theories in finance: Linear risk

- Capital Asset Pricing Model and Arbitrage Pricing Theory

- Bigger universe of nonlinear risk: not well-explored!

- Examples: Derivative securities and hedged funds

- Current Practice: Traders use Greek Letters, such as Delta, Gamma, Rho, ..., etc.

Main Idea: Boyle, Siu and Yang (2002)

- Consider a discrete-time financial model consisting of a risk-free bond B and a stock S

- Deal with a European call option C written on S with strike price K and maturity T

- Build the two-level binomial model from the CRR binomial model

- Evaluate a coherent risk measure, namely Expected Shortfall (ES), for derivative securities

The Model

- Suppose $\{0, 1, 2, \ldots, T\}$ is the time parameter set in the first level

- For each time point k in the first level, $[k, k+1]$ is the time interval for risk measurement

- Divide $[k, k+1]$ into m equal sub-intervals

- Then $\{0, 1, 2, \ldots, km, km+1, \ldots, Tm\}$ is the time parameter set in the second level

- For each sub-interval $[n, n+1]$ in the second level, assume that, under a real-world probability measure $\mathcal{P}$,

$$\frac{B_{n+1}}{B_n} = \widehat{r}$$

$$\frac{S_{n+1}}{S_n} = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p \end{cases}$$

- Call price from the CRR binomial model:

$$C_{km} = \frac{1}{\widehat{r}^{Tm-km}} \sum_{j=0}^{Tm-km} \binom{Tm-km}{j} q^j (1-q)^{Tm-km-j} (S_{km} u^j d^{Tm-km-j} - K)^+$$

Expected Shortfall (ES) for the Call

- $\Delta C_{k,m}$: the discounted net loss $C_{km} - \widehat{r}^{-m} C_{(k+1)m}$ of the call option C over $[km, (k+1)m]$

- $\mathcal{F}_{km}$: the information generated by the values of S up to and including time km

- Under $\mathcal{P}$, the distribution of $\Delta C_{k,m} | \mathcal{F}_{km}$:

$$\Delta C_{k,m} = C_{km} - \widehat{r}^{-m} C_{(k+1)m}(S_{km} u^j d^{m-j})$$

$$\text{with probability} \binom{m}{j} p^j (1-p)^{m-j} , \quad j = 0, 1, \ldots, m .$$

- ES for the call C:

$$
\begin{aligned}
& ES_{\alpha}(\Delta C_{k,m}|\mathcal{F}_{km}) \\
= \; & E_{\mathcal{P}}(\Delta C_{k,m}|I_{\{\Delta C_{k,m} \geq VaR_{\alpha}\}}, \mathcal{F}_{km}) \\
= \; & \alpha^{-1}[E_P(\Delta C_{k,m} I\{\Delta C_{k,m} \geq \mathrm{VaR}_{\alpha,P}(\Delta C_{k,m}|\mathcal{F}_{km})\}| \\
& \mathcal{F}_{km}) + \mathrm{VaR}_{\alpha,P}(\Delta C_{k,m}|\mathcal{F}_{km})(\alpha - P(\Delta C_{k,m} \geq \\
& \mathrm{VaR}_{\alpha,P}(\Delta C_{k,m}|\mathcal{F}_{km})|\mathcal{F}_{km})]
\end{aligned}
$$

- Adjustment for the discrete loss distribution to ensure the coherent property for the ES

An Expression for the ES

- Define $j_\alpha = \sup\{j \in J \mid \Delta C_{k,m}(j) \geq \mathrm{VaR}_{\alpha,P}(\Delta C_{k,m}|\mathcal{F}_{km})\}$, where J represents the set $\{0, 1, 2, \ldots, m\}$. Then

$$
\begin{aligned}
& ES_\alpha(\Delta C_{k,m}|\mathcal{F}_{km}) \\
= \; & -\frac{1}{\widehat{r}^{Tm-km}\alpha}\Bigg\{ \sum_{j=0}^{j_\alpha} \binom{m}{j} p^j (1-p)^{m-j} \Bigg[\sum_{i=0}^{Tm-(k+1)m} \binom{Tm-(k+1)m}{i} \\
& q^i (1-q)^{Tm-(k+1)m-i} (S_{km} u^{j+i} d^{Tm-km-j-i} - K)^+ \Bigg] - \sum_{i=0}^{Tm-km} \binom{Tm-km}{i} \\
& q^i (1-q)^{Tm-km-i} (S_{km} u^i d^{Tm-km-i} - K)^+ \Bigg\} \\
& + \Delta C_{k,m}(u^{j_\alpha} d^{m-j_\alpha}) \left[1 - \alpha^{-1} \sum_{j=0}^{j_\alpha} \binom{m}{j} p^j (1-p)^{m-j} \right].
\end{aligned}
$$

Numerical Example

- Consider a European call with $T = 2$ months and $K = 22$. Suppose $S_0 = 25$

- Assume that the time horizon for measuring the risk of the position is one month

- $r = 0.7\%$ per month and $\sigma = 6\%$ per month

- Two-level binomial model: $m = 5$, $u = e^{0.0268}$, $d = e^{-0.0268}$ and $q = 0.5194$

- The numerical values of ES and VaR for the call

Table: ES and VaR for various values of p and α.

$p \setminus \alpha$	0.01	0.05
0.3	2.795653 (2.795653)	2.795653 (2.795653)
0.4	2.795653 (2.795653)	2.795653 (2.795653)
0.5	2.795653 (2.795653)	2.49328 (1.989324)
0.6	2.795653 (2.795653)	2.15446 (1.989324)
0.7	2.185262 (1.989324)	1.591582 (0.852669)

Yang-Yin Grows Everything

- Yang-Yin generates many patterns

- Think about the modern computing technologies

- Central Limit Theorem: Binomial => Normal

- CRR binomial model => Continous-time Black-Scholes-Merton model

Risk Measures for Derivatives in Continuous-Time Markets

- Literature: Siu and Yang (2000) and Yang and Siu (2001), Siu, Tong and Yang (2002) and Elliott, Siu and Chan (2008)

- Siu and Yang (2000) and Elliott, Siu and Chan (2008): Use of stochastic optimal control theory to evaluate risk measures for derivatives

- "Bang-Bang" type control: Use in Aerospace engineering

- Paul Wilmott's book on Quantitative Finance and uncertain volatility models widely used in the finance industry

Risk Measures in Elliott, Siu and Chan (2008)

- Consider a financial model consisting of a bank account B and a share S

- A continuous-time, N-state observable Markov chain $\{\mathbf{X}(t)\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$.

- The price dynamics for B and S under $\mathcal{P}$:

$$\begin{aligned} dB(t) &= rB(t)dt \ , \\ dS(t) &= \mu(t)S(t)dt + \sigma(t)S(t)dW(t) \ , \end{aligned}$$

  where $\mu(t) := \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle$ and $\sigma(t) := \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle$; $\boldsymbol{\mu} := (\mu_1, \mu_2, \cdots, \mu_N)'$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \cdots, \sigma_N)'$.

First Step: Valuation

- Esscher transform: Esscher (1932), Gerber and Shiu (1994), Siu, Tong and Yang (2004) and Elliott, Chan and Siu (2005)

- The regime-switching Esscher transform by Elliott, Chan and Siu (2005):

  1. Define a process $\theta := \{\theta(t)\}$ by: $\theta(t) = \langle \boldsymbol{\theta}, \mathbf{X}(t) \rangle$, where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)'$.

  2. The regime-switching Esscher transform $\mathcal{Q}_\theta \sim \mathcal{P}$ associated with $\theta := \{\theta(t)\}$:

$$\left.\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\right|_{\mathcal{G}(t)} := \frac{\mathrm{exp}(\int_0^t \theta(u) dW(u))}{\mathrm{E}[\mathrm{exp}(\int_0^t \theta(u) dW(u)) | F^{\mathbf{X}}(t)]} \ .$$

- Consider a European-style option with payoff $V(S(T))$ at maturity T

- Given $S(t) = s$ and $\mathbf{X}(t) = \mathbf{x}$, a conditional price of the option is given by:

$$V(t, s, \mathbf{x}) = \mathsf{E}^{\theta}[e^{-r(T-t)} V(S(T)) | S(t) = s, \mathbf{X}(t) = \mathbf{x}] \ .$$

- **Proposition 1:** Let $V_i := V(t, s, \mathbf{e}_i)$, for each $i = 1, 2, \cdots, N$, and write $\mathbf{V} := (V_1, V_2, \cdots, V_N)' \in \Re^N$. Write $\mathbf{A}(t)$ for the rate matrix of the chain at time t. Then, V_i, $i = 1, 2, \cdots, N$, satisfy the following system of N-coupled P.D.E.s:

$$-rV_i + \frac{\partial V_i}{\partial t} + rs\frac{\partial V_i}{\partial s} + \frac{1}{2}\sigma_i^2 s \frac{\partial^2 V_i}{\partial s^2} + \langle \mathbf{V}, \mathbf{A}(t)\mathbf{e}_i \rangle = 0 \ ,$$

with terminal conditions $V(T, s, \mathbf{e}_i) = V(S(T))$, $i = 1, 2, \cdots, N$.

Second Step: Risk Evaluation

- For each $i = 1, 2, \cdots, N$, let $\Lambda_i = [\lambda_i^-, \lambda_i^+]$. For example, when $N = 2$ (i.e. State 1 is "Good Economy" and State 2 is "Bad Economy"), $\lambda_1^- = 0.05$; $\lambda_1^+ = 0.10$; $\lambda_2^- = 0.01$; $\lambda_2^+ = 0.05$.

- Suppose $\lambda(t)$ is the subjective appreciation rate of the share at time t. The chain modulates $\lambda(t)$ as:

$$\lambda(t) = \langle \boldsymbol{\lambda}, \mathbf{X}(t) \rangle \ ,$$

  where $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \cdots, \lambda_N)' \in \Re^N$ with $\lambda_i \in \Lambda_i$, $i = 1, 2, \cdots, N$.

- Consider, for each $\lambda \in \Theta$, a process $\{\theta^\lambda(t)\}$ defined by putting

$$\theta^\lambda(t) = \sum_{i=1}^{N} \left(\frac{\mu_i - \lambda_i}{\sigma_i}\right) \langle \mathbf{X}(t), \mathbf{e}_i \rangle \ .$$

- The regime-switching Esscher transform $\mathcal{P}_{\theta^\lambda} \sim \mathcal{P}$ on $\mathcal{G}(t)$ with respect to $\{\theta^\lambda(t)\}$:

$$\left.\frac{d\mathcal{P}_{\theta^\lambda}}{d\mathcal{P}}\right|_{\mathcal{G}(t)} := \frac{\exp(\int_0^t \theta^\lambda(u) dW(u))}{\mathsf{E}[\exp(\int_0^t \theta^\lambda(u) dW(u)) | F^{\mathbf{X}}(t)]} \ .$$

- Under $\mathcal{P}_{\theta^\lambda}$,

$$dS(t) = \lambda(t) S(t) dt + \sigma(t) S(t) dW^\lambda(t) \ ,$$

where $\{W^\lambda(t)\}$ is a $(\mathcal{G}, \mathcal{P}_{\theta^\lambda})$-standard Brownian motion.

- Future net loss of the option position over $[t, t+h]$:

  $$\Delta V(t,h) := e^{rh} V(t, S(t), \mathbf{X}(t)) - V(t+h, S(t+h), \mathbf{X}(t+h))$$

- Given $S(u) = s$ and $\mathbf{X}(u) = \mathbf{x}$, $u \in [t, t+h]$, the generalized scenario expectation for the option position V over $[t, t+h]$:

  $$\begin{aligned} & \rho(u, s, \mathbf{x}) \\ := \ & \sup_{\lambda \in \Theta} \mathsf{E}^{\theta^\lambda}[\exp(-r(t+h-u)) \Delta V(t,h) | S(u) = s, \mathbf{X}(u) = \mathbf{x}] \ , \end{aligned}$$

  where $\mathsf{E}^{\theta^\lambda}[\cdot]$ is an expectation under $\mathcal{P}_{\theta^\lambda}$.

- Write $\rho_i := \rho(u, s, \mathbf{e}_i)$, $i = 1, 2, \cdots, N$, and $\boldsymbol{\rho} := (\rho_1, \rho_2, \cdots, \rho_N)'$.

- **Proposition 2.** For each $i = 1, 2, \cdots, N$, let $\Delta_i^R := \frac{\partial \rho_i}{\partial s}$ and $\lambda(\Delta_i^R) = \begin{cases} \lambda_i^+ & \text{if } \Delta_i^R > 0 \\ \lambda_i^- & \text{if } \Delta_i^R < 0 \end{cases}$. Then ρ_i, $i = 1, 2, \cdots, N$, satisfy the following system of N-coupled P.D.E.s:

$$\frac{\partial \rho_i}{\partial u} + \frac{1}{2}\sigma_i^2 s^2 \frac{\partial^2 \rho_i}{\partial s^2} + \lambda(\Delta_i^R) s \frac{\partial \rho_i}{\partial s} - r\rho_i + \langle \boldsymbol{\rho}, \mathbf{A}(t)\mathbf{e}_i \rangle = 0 \ ,$$

with the following terminal conditions:

$$\rho(t+h, S(t+h), \mathbf{e}_i) = e^{rh} V(t, S(t), \mathbf{X}(t)) - V(t+h, S(t+h), \mathbf{e}_i) \ .$$

- For the case of an American-style option, a system of coupled variational inequalities for the risk measures was obtained.

What Next?

- Incorporate credit risk and counterparty risk in the OTC markets

- Liquidity risk due to large trading positions

- Applications to modern insurance products with embedded options

- Non-Markovian situation: Use of functional Itôś calculus for nonlinear evaluation of dynamic convex risk measures in Siu (2011)

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