Capital allocation and diversification benefits

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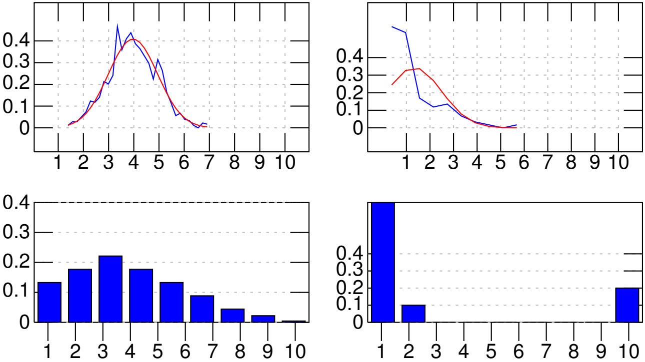
Possible Loss Distributions



- How much should we set aside to cover these potential losses?
- Mean + z standard deviations not such a good idea
- What if losses/risks are "correlated"?
- What is a good measure of "risk"?

"Normal" calculations

- For each loss $\boxed{y_i}$ set aside the mean plus a risk margin $\boxed{\dot{y}_i}$
- $\boxed{\dot{y}_i}$ could be $\boxed{z\sigma_i}$ where z is appropriately chosen
- V@R risk margin is then $2\sigma_i$ (95%), $3\sigma_i$ (99%), ... ?
- Each margin $\dot{y}_i$ is determined relative to its own distribution and (here) called the "standalone" risk margin
- Total standalone margin is then $\boxed{\sum_i \dot{y}_i = z \sum_i \sigma_i}$
- This method of determining risk margins is terrible in many financial (non normal) situations
  - σ_i is/can be terrible measure of "spread/risk" if loss distribution not normal
  - V@R is often a terrible risk measure
  - Losses may be "correlated"
  - No allowance for "diversification"

"Normal" calculations allowing for correlation

- Total loss $y \equiv \sum_i y_i$
- Standard deviation of y is

$$\sigma_y = \sqrt{\sum_{i,j} \sigma_{ij}} = \sqrt{\sum_{i,j} \sigma_i \sigma_j \rho_{ij}}$$

- Risk margin on total is then $\boxed{\dot{y} = z\sigma_y}$
- $\boxed{\dot{y}}$ is the risk margin for y based on distribution of y
- With V@R, $z = 2$ for 95%, $z = 3$ for 99%, ...
- $\boxed{\dot{y} \neq \sum \dot{y}_i}$ where $\dot{y}_i$ is the "standalone" risk margin for risk i even if losses/risks are independent.
- But ...
  - How to attribute $\dot{y}$ back to the individual losses y_i?
  - How do we measure correlations? (subjective, positive definite)
  - Correlation is often a <u>terrible</u> measure of dependence
  - Still basically using (yuck) V@R

Summarize

If losses/risks are not normal then standard deviation/correlation type calculations to calculate total or individual risk margins appear deeply flawed.

- Let's throw it all out ... including, while at it, V@R
- I argue

"throwing it all out"

≡ "throwing out the baby with the bathwater"

- What is the "bathwater" here and what is the "baby"?
  - "Bathwater" – normal, V@R
  - "Baby" – standard deviation, correlation, covariance
- Actually V@R is "knocked down, picked up and brushed off"

Risk measures

- Let $u_i \equiv F_i(y_i)$ where F_i is the distribution of y_i
- Thus u_i is the percentile associated with y_i
- Critical Result 1: Any (coherent) risk measure is of the form

$$\boxed{\text{“risk”} = \mathrm{E}\{y_i\phi(u_i)\} = \mathrm{E}(y_i) + \sigma_\phi\rho_i\sigma_i}$$

  where $\mathrm{E}\{\phi(u_i)\} = 1$ and $\phi \geq 0$ is nondecreasing
  - V@R: ϕ is zero unless eg $u_i = 0.95$ when it is ∞ ... ??
  - CTE: ϕ is a step function: 0 unless eg $u_i > 0.75$ ... expectation in tail
  - Emax: $\phi(u_i) = c(1 - u_i^n)$ ... worst of n outcomes
  - ... minmax, maxmin, ... (see papers)
- Hence any risk margin based on any risk measure is of the form

$$\boxed{\dot{y}_i \equiv \sigma_\phi\rho_i\sigma_i} \quad \ldots \quad \text{spot the baby}$$

Risk margin on total $y = \sum_i y_i$

- Let $u = F(y)$ where F is the distribution of $y \equiv \sum_i y_i$
- Thus u is the percentile associated with y
- As before, $\boxed{\text{“risk”} = \mathrm{E}\{y\phi(u)\} = \mathrm{E}(y) + \sigma_\phi \rho_y \sigma_y}$
- Thus risk margin on total is $\dot{y} \equiv \sigma_\phi \rho_y \sigma_y$
- Critical (but trivial) Result 3: (... spot the baby)

$$\boxed{\dot{y} \equiv \mathrm{cov}\{y, \phi(u)\} = \sum_i \mathrm{cov}\{y_i, \phi(u)\} = \sigma_\phi \sum_i \sigma_i \rho_i^+}$$

- $\boxed{\ddot{y}_i \equiv \sigma_\phi \sigma_i \rho_i^+}$ are called the “stand together” or “diversified” risk margin and compared to $\boxed{\dot{y}_i \equiv \sigma_\phi \sigma_i \rho_i}$
- Natural capital allocation formula $\boxed{\dot{y} = \sum_i \ddot{y}_i}$!

Diversification benefit

- $\boxed{\dot{y}_i \equiv \sigma_\phi \sigma_i \rho_i}$ versus $\boxed{\ddot{y}_i \equiv \sigma_\phi \sigma_i \rho_i^+}$
- $\boxed{\sum_i \dot{y}_i \neq \sum_i \ddot{y}_i = \dot{y}}$
- Diversification benefit:

$$\sum_i (\dot{y}_i - \ddot{y}_i) = \sum_i \dot{y}_i \left(1 - \frac{\ddot{y}_i}{\dot{y}_i}\right) = \sum_i \dot{y}_i \left(1 - \frac{\rho_i^+}{\rho_i}\right)$$

- Percentage saved: Divide above by $\sum_i \dot{y}_i$

$$\sum_i \left(\frac{\dot{y}_i}{\sum_i \dot{y}_i}\right) \left(1 - \frac{\rho_i^+}{\rho_i}\right)$$

- In words: Percentage saved is the weighted average of individual diversification benefits $1 - \rho_i^+/\rho_i$ where the weights are the fractions of standalone margin

Summarize

- For any risk measure $\boxed{\dot{y}_i = \sigma_\phi \sigma_i \rho_i}$ ... standalone risk margin
- Risk margin on total $\boxed{\dot{y} = \sigma_\phi \sum_i \sigma_i \rho_i^+}$
- Risk margin capital $\boxed{\dot{y}}$ allocated according to $\boxed{\sigma_\phi \sigma_i \rho_i^+}$
- Percentage saved = Weighted average of $\boxed{1 - \rho_i^+ / \rho_i}$
- Weights proportional to standalone margins $\boxed{\dot{y}_i}$
- σ_ϕ is fixed in all this ... "conservativism" factor
- σ_i is standard deviation of risk i – "easy" to determine.
- Correlations ρ_i and ρ_i^+, the correlation between risk y_i and $\phi(u_i)$ and $\phi(u)$ more problematic (copulas etc)
- Thus means, standard deviations and correlations can still be used! Albeit in a different metric
- $\boxed{\text{The baby (babies?) has (have) been saved}}$!!!

Example (from insurance)

- Expected: $\boxed{\mathrm{E}(y_i)}$, standalone: $\boxed{\dot{y}_i}$, diversified: $\boxed{\ddot{y}_i}$

line of business	expected	stand alone	diversi¯ed
1	372.41	21.40	17.03
2	96.49	8.97	0.46
3	116.05	15.74	13.05
4	254.41	15.98	10.97
5	289.75	15.30	10.45
6	6.54	0.96	0.48
7	5.06	1.49	0.13
8	2.23	0.46	0.01
total	1142.93	80.30	52.57

Example (cont)

- Blue: $\boxed{1 - \rho_i^+/\rho_i}$, Red: $\boxed{c\dot{y}_i}$, Green: Blue $\times$ Red