A General Theory of Backward Stochastic Difference Equations

Samuel N. Cohen and Robert J. Elliott

University of Adelaide and University of Calgary

Outline

Risk and Pricing

- A key question in Mathematical Finance is: *Given a future **random** payoff X, what are you willing to pay **today** for X?*
- One could also ask *"How **risky** is X?"*
- Various attempts have been made to answer this question. (Expected utility, CAPM, Convex Risk Measures, etc...)
- While giving an axiomatic approach to answering this question, we shall outline the theory of "Backward Stochastic Difference Equations."

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Nonlinear Expectations

For some terminal time T, we define an '$\mathcal{F}_t$-consistent nonlinear expectation' $\mathcal{E}$ to be a family of operators

$$\mathcal{E}(\cdot|\mathcal{F}_t) : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t); t \leq T$$

with

1. (Monotonicity) If $Q^1 \geq Q^2$ $\mathbb{P}$-a.s.,

$$\mathcal{E}(Q^1|\mathcal{F}_t) \geq \mathcal{E}(Q^2|\mathcal{F}_t)$$

2. (Constants) For all $\mathcal{F}_t$-measurable Q,

$$\mathcal{E}(Q|\mathcal{F}_t) = Q$$

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Nonlinear Expectations

3. (Recursivity) For $s \leq t$,

$$\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s)$$

4. (Zero-One law) For any $A \in \mathcal{F}_t$,

$$\mathcal{E}(I_A Q|\mathcal{F}_t) = I_A \mathcal{E}(Q|\mathcal{F}_t).$$

Nonlinear Expectations

Two other properties are desirable

5. (Translation invariance) For any $q \in L^2(\mathcal{F}_t)$,

$$\mathcal{E}(Q + q|\mathcal{F}_t) = \mathcal{E}(Q|\mathcal{F}_t) + q.$$

6. (Concavity) For any $\lambda \in [0, 1]$,

$$\mathcal{E}(\lambda Q^1 + (1 - \lambda)Q^2|\mathcal{F}_t) \geq \lambda\mathcal{E}(Q^1|\mathcal{F}_t) + (1 - \lambda)\mathcal{E}(Q^2|\mathcal{F}_t)$$

Nonlinear Expectations

There is a relation between nonlinear expectations and convex risk measures:

- If (1)-(6) are satisfied, then for each t,

$$\rho_t(X) := -\mathcal{E}(X|\mathcal{F}_t)$$

  defines a dynamic convex risk measure. These risk measures are *time consistent*.
- For simplicity, this presentation will discuss nonlinear expectations.
- How could we construct such a family of operators?

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- In this presentation, we shall consider discrete time processes satisfying ‘Backward Stochastic Difference Equations’.
- These are the natural extension of Backward Stochastic Differential Equations in continuous time.
- We shall see that *every* nonlinear expectation satisfying Axioms (1-5) solves a BSDE with certain properties, and conversely.
- We also establish necessary and sufficient conditions for concavity (Axiom 6).
- To do this, we first need to set up our probability space.

A probabilistic setting

- Let X be a discrete time, finite state process. Without loss of generality, X takes values from the unit vectors in $\mathbb{R}^N$.
- Let $\{\mathcal{F}_t\}$ be the filtration generated by X, that is $\mathcal{F}_t$ consists of every event that can be known from watching X up to time t.
- Let $M_t = X_t - E[X_t|\mathcal{F}_{t-1}]$. Then M is a martingale difference process, that is $E[M_t|\mathcal{F}_{t-1}] = \mathbf{0} \in \mathbb{R}^N$.

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Discrete BSDEs ('D=Difference')

A BSDE is an equation of the form:

$$Y_t - \sum_{t \leq u < T} F(\omega, u, Y_u, Z_u) + \sum_{t \leq u < T} Z_u M_{u+1} = Q$$

- Q is the terminal condition (in $\mathbb{R}$)
- F is a (stochastic) 'driver' function, with $F(\omega, u, \cdot, \cdot)$ known at time u.
- A solution is an adapted pair (Y, Z) of processes, $Y_t \in \mathbb{R}$ and $Z_t \in \mathbb{R}^{N \times 1}$.
- All quantities are assumed to be $\mathbb{P}$-a.s. finite.

Discrete BSDEs ('D=Difference')

Equivalently, we can write this in a differenced form:

$$Y_t - F(\omega, t, Y_t, Z_t) + Z_t M_{t+1} = Y_{t+1}$$

with terminal condition

$$Y_T = Q.$$

The important detail is that

- The *terminal* condition is fixed, and the dynamics are given in reverse.
- The solution (Y, Z) is adapted, that is, at time t it depends only on what has happened up to time t.

A Special Case: Binomial Pricing

- Suppose we have a market with two assets: a stock Y following a simple binomial price process, and a risk free Bond B.
- Let r_t denote the one-step interest rate at time t.
- From each time t, there are two possible states for the stock price the following day, $Y(t+1,\uparrow)$ and $Y(t+1,\downarrow)$.
- Suppose these two states occur with (real world) probabilities p and $1-p$ respectively.

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It is easy to show that there exists a unique 'no-arbitrage' price

$$
\begin{aligned}
Y(t) &= \frac{1}{1+r_t}\left[\pi Y(t+1,\uparrow) + (1-\pi)Y(t+1,\downarrow)\right] \\
&= \frac{1}{1+r_t} E_\pi[Y(t+1)|\mathcal{F}_t].
\end{aligned}
$$

Here π the 'risk-neutral probability', that is, the price today is the average discounted price tomorrow; when π is the probability of a price increase.

Writing $Y_t = Y(t)$ etc... we also know,

$$Y_{t+1} = E_p(Y_{t+1}|\mathcal{F}_t) + L_{t+1}$$

where $E_p(Y_{t+1}|\mathcal{F}_t)$ is the (real-world) conditional mean value of Y_{t+1}, and L_{t+1} is a random variable with conditional mean value zero

$$(L_{t+1} = Y_{t+1} - E_p(Y_{t+1}|\mathcal{F}_t)).$$

In the notation we established before, we can define a martingale difference process M

$$M_{t+1}(\uparrow) = \begin{bmatrix} 1-p \\ p-1 \end{bmatrix}, M_{t+1}(\downarrow) = \begin{bmatrix} -p \\ p \end{bmatrix}.$$

And it is easy to show that L_{t+1} can be written as $Z_t M_{t+1}$, for some row vector Z_t known at time t. (Doob-Dynkin Lemma)

We can then do some basic algebra:

$$\begin{aligned}
Y_{t+1} &= E_p(Y_{t+1}|\mathcal{F}_t) + L_{t+1} \\
&= Y_t + r_t Y_t - (1+r_t)Y_t + E_p(Y_{t+1}|\mathcal{F}_t) + Z_t M_{t+1} \\
&= Y_t + r_t Y_t - (1+r_t)\frac{1}{1+r_t}E_\pi(Y_{t+1}|\mathcal{F}_t) + E_p(Y_{t+1}|\mathcal{F}_t) + Z_t M_{t+1} \\
&= Y_t + r_t Y_t - E_\pi(Y_{t+1} - E_p(Y_{t+1})|\mathcal{F}_t) + Z_t M_{t+1} \\
&= Y_t + r_t Y_t - E_\pi(L_{t+1}|\mathcal{F}_t) + Z_t M_{t+1} \\
&= Y_t - \Big[- r_t Y_t + Z_t E_\pi(M_{t+1}|\mathcal{F}_t)\Big] + Z_t M_{t+1} \\
&= Y_t - F(Y_t, Z_t) + Z_t M_{t+1}
\end{aligned}$$

So our one-step pricing formula is equivalent to the equation

$$Y_{t+1} = Y_t - F(Y_t, Z_t) + Z_t M_{t+1}$$

where

$$\begin{aligned} F(Y_t, Z_t) &= -r_t Y_t + Z_t E_\pi(M_{t+1} | \mathcal{F}_t) \\ &= -r_t Y_t + Z_t \begin{bmatrix} \pi - p \\ p - \pi \end{bmatrix} \end{aligned}$$

This is a special case of a BSDE.

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Before giving general existence properties of BSDEs, we need the following.

Definition

If $Z_t^1 M_{t+1} = Z_t^2 M_{t+1}$ $\mathbb{P}$-a.s. for all t, then we write $Z^1 \sim_M Z^2$. Note this is an equivalence relation for $Z_t \in \mathbb{R}^{N\times 1}$.

Theorem

For any $\mathcal{F}_{t+1}$-measurable random variable $W \in \mathbb{R}$ with $E[W|\mathcal{F}_t] = 0$, there exists a $\mathcal{F}_t$-measurable $Z_t \in \mathbb{R}^{N\times 1}$ with

$$W = Z_t M_{t+1}.$$

An Existence Theorem

Theorem

Suppose

(i) $F(\omega, t, Y_t, Z_t)$ *is invariant under equivalence* $\sim_M$

(ii) *For all* Z_t*, the map*

$$Y_t \mapsto Y_t - F(\omega, t, Y_t, Z_t)$$

is a bijection

Then a BSDE with driver F *has a unique solution in* L^1.

Corollary

These conditions are necessary and sufficient.

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Proof:

Let $Z_t \in \mathbb{R}^{N\times 1}$ solve

$$Z_t M_{t+1} = Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t].$$

Then let $Y_t \in \mathbb{R}$ solve

$$Y_t - F(\omega, t, Y_t, Z_t) = E[Y_{t+1}|\mathcal{F}_t]$$

for the above value of Z_t.
Then (Y_t, Z_t) solves the one step equation

$$Y_t - F(\omega, t, Y_t, Z_t) + Z_t M_{t+1} = Y_{t+1},$$

and the result follows by backwards induction.

- We wish to ensure that, when $Q^1 \geq Q^2$, the corresponding values $Y_t^1 \geq Y_t^2$ for all t.
- This will, (eventually), allow us to define a nonlinear expectation $\mathcal{E}$ and obtain the monotonicity and concavity assumptions.
- The key theorem here is the *Comparison Theorem*

Definition

We define $\mathbb{J}_t$, the set of possible jumps of X from time t to time $t+1$, by

$$\mathbb{J}_t := \{i : \mathbb{P}(X_{t+1} = e_i | \mathcal{F}_t) > 0)\}.$$

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Comparison Theorem

Theorem

Consider two BSDEs with drivers F^1, F^2, terminal values Q^1, Q^2, etc... Suppose that, $\mathbb{P}$-a.s. for all t,

(i) $Q^1 \geq Q^2$

(ii) $F^1(\omega, t, Y_t^2, Z_t^2) \geq F^2(\omega, t, Y_t^2, Z_t^2)$

(iii) $F^1(\omega, t, Y_t^2, Z_t^1) - F^1(\omega, t, Y_t^2, Z_t^2) \geq \min_{j \in \mathbb{J}_t}\{(Z_t^1 - Z_t^2)(e_j - E[X_{t+1}|\mathcal{F}_t])\}.$

(iv) *The map $Y_t \mapsto Y_t - F(\omega, t, Y_t, Z_t^1)$ is strictly increasing in Y_t.*

Then $Y_t^1 \geq Y_t^2$ $\mathbb{P}$-a.s. for all t.

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Proof:

Assume $Y^1_{t+1} - Y^2_{t+1} \geq 0$, then, omitting ω and t,

$$\begin{aligned} & Y^1_t - Y^2_t - F^1(Y^1_t, Z^1_t) + F^1(Y^2_t, Z^1_t) \\ & \quad - [F^1(Y^2_t, Z^1_t) - F^1(Y^2_t, Z^2_t)] + (Z^1_t - Z^2_t)M_{t+1} \\ & = [Y^1_{t+1} - Y^2_{t+1}] + [F^1(Y^2_t, Z^2_t) - F^2(Y^2_t, Z^2_t)] \\ & \geq 0. \end{aligned}$$

This must hold $\mathbb{P}$-a.s., so it holds under taking the $\mathcal{F}_t$-conditional essential minimum of all terms. Hence

$$Y^1_t - Y^2_t - F^1(Y^1_t, Z^1_t) + F^1(Y^2_t, Z^1_t) \geq 0$$

and then as $Y_t \mapsto Y_t - F(Y_t, Z^1_t)$ is strictly increasing, the result follows by induction.

- Given this theory, we can now construct explicit examples of nonlinear expectations.
- In fact, every nonlinear expectation can be constructed this way.

BSDEs and Nonlinear Expectations

Theorem

The following statements are equivalent:

(i) $\mathcal{E}(\cdot|\mathcal{F}_t)$ *is an* $\mathcal{F}_t$*-consistent, translation invariant nonlinear expectation. (Axioms 1-5)*

(ii) *There is an* F *such that* $Y_t = \mathcal{E}(Q|\mathcal{F}_t)$ *solves a BSDE with driver* F *and terminal condition* Q*, where* F *satisfies the conditions of the comparison theorem, is independent of* Y_t*, and* $F(\omega, t, Y_t, 0) = 0$ $\mathbb{P}$*-a.s. for all* t*.*

In this case,

$$F(\omega, t, Y_t, Z_t) = \mathcal{E}(Z_t M_{t+1}|\mathcal{F}_t).$$

Corollary

The nonlinear expectation $\mathcal{E}(\cdot|\mathcal{F}_t)$ has property '...' if and only if F has property '...' (in Z), where '...' is any of:

- *Concavity*
- *Positive homogeneity*
- *Linearity*
- *Invariance under addition of martingale terms orthogonal to a given process*
- *(Lipshitz) continuity (in L^1 norm)*
- *etc...*

Note, these statements are trivial, given the equivalence

$$F(\omega, t, Y_t, Z_t) = \mathcal{E}(Z_t M_{t+1}|\mathcal{F}_t).$$

- The proof of this is simple, but long.
- This result holds for both scalar and vector valued nonlinear expectations.
- Similar results have been obtained for the scalar Brownian Case, (Coquet et al, 2002), (Hu et al, 2008).
- In discrete time everything is simpler, and one can even obtain similar results for the more general nonlinear evaluations (Cohen & Elliott, forthcoming)

An Example

To demonstrate the complexity that can be achieved, consider a two step world where X_t takes one of two values with equal probability.
Assume that Z_t is written in the form $Z_t = [z, -z]$, which is unique up to equivalence $\sim_{M_t}$. We consider the concave function

$$F(\omega, t, Y_t, Z_t) = \min_{\pi \in [0.1, 0.9]} \{2(\pi - 0.5)z + \gamma(\pi - 0.5)^2\},$$

where γ is a 'risk aversion' parameter (the smaller the value of γ, the more risk averse), which we shall set to $\gamma = 10$.

Other Results:

- One can show under what conditions a generic monotone map $L^2(\mathcal{F}_T) \to \mathbb{R}$ can be extended to an $\mathcal{F}_t$ consistent nonlinear expectation.
- It is also possible, in general, to determine under what conditions the driver F can be determined from the solutions Y_t, even when the comparison theorem and normalisation conditions do not hold.
- These results are significantly stronger than available in continuous time.

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Conclusions

- The theory of BSDEs can be expressed in discrete time.
- Various continuous time results, such as the comparison theorem, extend naturally to the discrete setting.
- The discrete time proofs are often simpler than in continuous time, and give stronger results.
- It forms a natural setting for nonlinear expectations, as every nonlinear expectation solves a BSDE.
- This has various implications for problems in economic regulation, and in other areas of optimal stochastic control.